Def'n: Given $\alpha \in C^p(X)$, $\beta \in C_q(X)$,

define $\alpha \frown \beta := \mathrm{id}\otimes\alpha\,(\underbrace{\Delta\beta})$ lives in $(C_\bullet(X)\otimes C_\bullet(X))_q := \bigoplus_{i+j=q} C_i(X)\otimes C_j(X)$

$\Delta\beta := \sum_{i+j=q} (\Delta\beta)_{i,j}$

recall $\alpha : C_p(X) \to R$,

extend to $\alpha : C_\bullet(X) \to R$ by saying $\alpha(C_i(X)) = 0$ for $i \neq p$.

i.e., $\alpha\frown\beta = \mathrm{id}\otimes\alpha\,(\Delta\beta_{q-p,p}) \in C_{q-p}(X)$,

b/c $\mathrm{id}\otimes\alpha$ is only non-zero on this piece.

If we use the Alexander-Whitney model of homological coproduct

$$\Delta_{AW} := \Theta_{AW}\circ\Delta_\# : \underset{\text{degree } q}{\sigma} \longmapsto \sum_{i+j=q} \sigma|_{[e_0,\dots,e_i]} \otimes \sigma|_{[e_i,\dots,e_{i+j=q}]},$$

front i-face, back j-face

we see that up to chain homotopy the cap product can be given the following chain-level model: linearly extend the following formula over all chains

$\exists$ a sign to apply α to this term

$$\alpha^p \frown \sigma_q := (\mathrm{id}\otimes\alpha^p)\left(\sigma|_{[e_0,\dots,e_{q-p}]} \otimes \sigma|_{[e_{q-p},\dots,e_{q-p+p=q}]}\right)$$

front $q-p$ face, back p face.

$\alpha^p \in C^p(X;R)$, $\sigma_q \in C_q(X;R)$ generator $\sigma : \Delta^q \to X$.

$$= (-1)^{p(q-p)}\, \sigma|_{[e_0,\dots,e_{q-p}]}\cdot \underbrace{\alpha(\sigma|_{[e_{q-p},\dots,e_q]})}_{\in R}.$$

Prop: For any Θ (not just Θ_{AW}), the cap product satisfies the property of being a chain map

$$C^{-\bullet}(X)\otimes C_\bullet(X) \xrightarrow{\frown} C_\bullet(X) \quad \text{(of degree 0)}.$$

co-chains on X w/ degrees negated is a chain complex, in sense that δ decreases ($-$degree) by 1.

$\underbrace{C^p(X)}\otimes C_q(X) \xrightarrow{\frown} C_{q-p}(X)$.

think of this as a degree $-p$ element of $C^{-\bullet}(X)$, then degrees are additive under cap product.

Namely, $\boxed{\partial(\alpha\frown\sigma) = \delta\alpha\frown\sigma + (-1)^p\,\alpha\frown\partial\sigma}$ (where $\alpha\in C^p(X)$).

Hence, there is an induced map $H^\bullet(X) \otimes H_\bullet(X) \xrightarrow{\cap} H_\bullet(X)$, which negating degrees of cohomology groups, is a graded map.

<u>Prop</u>: $\cap$ on homological level is independent of choice of Θ (not hard to see), and satisfies:

(1) $\underset{\substack{\cap \\ H^0(X)}}{1} \cap \gamma = \gamma$ for all $\gamma \in H_p(X)$

(2) If $\varepsilon : X \to pt$ induces $\varepsilon_* : H_0(X) \xrightarrow{\cong} H_0(pt) = R$ ('augmentation') (note $1 = \varepsilon^*(1)$), and $[\alpha] \in H^p(X)$, $[\tau] \in H_p(X)$, then $\underset{\substack{\cap \\ R}}{\varepsilon_*}(\underbrace{[\alpha] \cap [\tau]}_{\substack{\cap \\ H_0(X)}}) = \alpha(\tau)$

$\alpha \in C^p(X;R) = Hom_R(C_p(X), R)$.

(3) $(\alpha \cup \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma)$

(note an action of R on M is a module action $\Rightarrow (r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$).

(4) If $\lambda : X \to Y$ is a map of top. spaces, $\alpha \in H^\bullet(Y)$, $\beta \in H_\bullet(X)$, then:

$$\underbrace{\alpha \cap \lambda_*(\beta)}_{\substack{\cap \\ H_\bullet(Y)}} = \lambda_*(\underbrace{\lambda^*\alpha}_{\substack{\cap \\ H^\bullet(X)}} \cap \beta).$$

<u>Interpretation</u>: Since $\lambda^* : H^\bullet(Y) \to H^\bullet(X)$ is a ring map, $H_\bullet(X)$ (a module over $H^\bullet(X)$) becomes via λ^* a module over $H^\bullet(Y)$ via the action "applying λ^* then $\cap$".

(4) is then stating that $H_\bullet(X) \xrightarrow{\lambda_*} H_\bullet(Y)$ is a map <u>of $H^\bullet(Y)$-modules</u> (w.r.t. this module action of $H^\bullet(Y)$ on $H_\bullet(X)$ induced by λ^*).

<u>Proofs of prop</u>: Straightforward: (4) is an immediate consequence of naturality (& holds on chain level for any Θ), and (1)-(3) can be checked on chain level for the Alexander-Whitney model of $\cap$, hence hold homologically for any model. (e.g., (3) follows from "Δ is co-associative") (<u>exercise</u>).

<u>Rmks</u>: If $A \subset X$, and c is a chain in A, then $\alpha \cap c$ is a chain in A too, for any $\alpha \in C^\bullet(X)$, so get $\cap : C^p(X) \otimes C_n(X, A) \longrightarrow C_{n-p}(X, A)$, inducing a homology level cap product.

$C^\bullet(X) \otimes C_\bullet(X) \to C_\bullet(X)$

$\cup$ $Ann(C_\bullet(A))$ $\parallel$

$Ann(C_\bullet(A)) \otimes C_\bullet(X) \to C_\bullet(X)$ sends $Ann(C_\bullet(A)) \otimes C_\bullet(A) \to 0$, hence get a map

<u>Exercises</u>:

- Also get

$C^p(X,A) \otimes C_n(X,A) \longrightarrow C_{n-p}(X)$ $\quad \mathrm{Ann}(C_\bullet(A)) \otimes \frac{C_\bullet}{C_\bullet(A)} \to C_\bullet(X)$,

idea: elements here are are zero on $C_\bullet(A) \subset C_\bullet(X)$, so unaffected by adding chains
chains on A.

• More generally, if $A, B \underset{\text{open}}{\subset} X$, $\left(C_\bullet(\text{“}A+B\text{”}) \simeq C_\bullet(A \cup B) \right)$,

means $C_\bullet(A) + C_\bullet(B)$ (in $C_\bullet(X)$).

get $C^p(X,A) \otimes C_n(X, \text{“}A+B\text{”}) \xrightarrow{\frown} C_{n-p}(X,B)$, inducing

meaning $\dfrac{C_n(X)}{C_n(A)+C_n(B)}$

$$H^p(X,A) \otimes H_n(X, A\cup B) \to H_{n-p}(X,B).$$

etc. .

<u>Orientations</u>:

On a finite-dim'l vector space V^n ($n := \dim(V)$), an orientation is a choice of ordered basis up to equivalence, where bases

$(v_1, \dots, v_n)$ and $(w_1, \dots, w_n)$ are equivalent if the map taking one to another has <u>positive</u> determinant.

To generalize to the case $n=0$ in a uniform way, we might equivalently say an orientation is a choice of non-zero element of $\Lambda^{\dim(V)} V$ (given $\Lambda^0 \{0\} \cong \mathbb{R}$ canonically), up to the equivalence relation of <u>positive</u> scaling.

$\Rightarrow$ two possible choices. (even in dim 0, where the two choices are $\{+,-\}$).

Denote by $o(V)$ (or $or(V)$) the set of orientations on V (two elements, but not canonically identified w/ $+/-$ unless dimension is 0. On the other hand the operation called "orientation reversal" induces a free $\mathbb{Z}/2$ action on $o(V)$ — so $o(V)$ is a $\mathbb{Z}/2$-<u>torsor</u>).

If M^n <u>smooth manifold</u>: (say $M \hookrightarrow \mathbb{R}^N$)

Then at any p, have a tangent space T_pM (vec space of dim. n), & can pick an orientation of T_pM, $o_p \in o(T_pM)$

An orientation on M is roughly a "continuously or smoothly" or "coherently" varying choice of such $\{o_p\}_{p\in M}$.

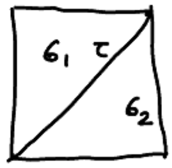
Say $\{o_p\}_{p\in M}$ is 'coherent' if $\forall p \in M$, $\exists$ $\mathcal{U} \ni p$ and a basis of vector fields $v_1, \dots, v_n$ over $\mathcal{U}$ with $o_q = o((v_1)_q, \dots, (v_n)_q)$ for any $q \in \mathcal{U}$.

(in particular, given a fixed o_p, coherence condition induces a unique o_q on any such $\mathcal{U}$).

—

How to generalize to the not nec. smooth case?

piece of M :



Idea: (old historical) Say M admits a fixed triangulation.

one attempt to 'orient' M is to 'orient each top simplex'.

(meaning order the vertices for each σ): so that (coherence) for any

σ_1, σ_2 sharing an edge τ, τ appears ~~up~~ w/ opposite signs in $\underline{\partial \sigma_1}$ and $\underline{\partial \sigma_2}$. ($\Rightarrow \tau$ cancels in $\partial(\sigma_1+\sigma_2)$).

(recall $\partial[e_0, \ldots, e_n] = \sum (-1)^i [e_0, \ldots, \hat{e_i}, \ldots, e_n]$, signs depend on orderings)

($\Rightarrow \partial(\sum \sigma_i) = 0$ if M cpct, so expect $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$), (connected).

top cycles

Problem: ad hoc, depends on triangulations.

Def: A (topological) manifold M of dimension n, denoted M^n is a space (implicitly Hausdorff, 2^{nd} countable), which is locally homeomorphic to $\mathbb{R}^n$.

(i.e., at each p $\exists$ open $U \ni p$ w/ $(U,p) \underset{\text{homeo}}{\cong} (\mathbb{R}^n, 0)$.) .

How to define a local orientation of M at x?

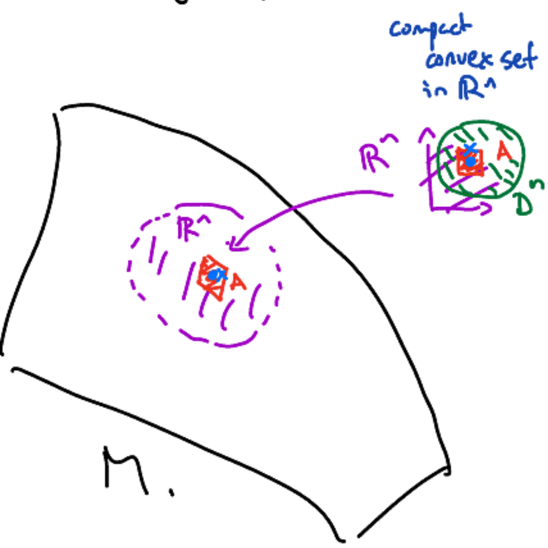
idea: an ordering of the vertices of this simplex should determine a 'local' orientation of M at x.

idea: such simplices live in $H_n(M, M-x)$, σ an ordering determines a choice of generator.

Lem: M^n manifold, $x \in M$ any point, R any coeff grp (imagine $\mathbb{Z}$ for now).

Then $H_n(M, M-\{x\}; R) \cong R$.

More generally, if $A \subset \mathbb{R}^n \xhookrightarrow{\text{open}} M$, then $H_n(M, M-A) \xrightarrow{\cong} H_n(M, M-x) \xrightarrow{\cong} R$. (for any $x \in A$)



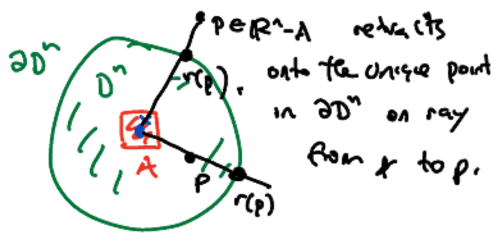
Pf: $\exists$ a closed ball $D^n \subset \mathbb{R}^n$ centered at x, containing A in its interior. Now, note there's a homotopy equivalence of pairs $D^n \xhookrightarrow{\sim} \mathbb{R}^n$ and $\partial D^n \xhookrightarrow[\text{incl.}]{\text{h.e.}} \mathbb{R}^n - A$

Q: why is this true? (exercise: need a retraction $\mathbb{R}^n - A \to \partial D^n$.

$$
\begin{tikzcd}
(D^n, \partial D^n) \arrow[r, hook, "\sim"', "\text{incl.}"] & (\mathbb{R}^n, \mathbb{R}^n - A) \arrow[dl, "\text{incl.}"] \\
(\mathbb{R}^n, \mathbb{R}^n - x)
\end{tikzcd}
$$

$(\text{for } x \in A) \Big\}$ incl. (htpy equiv. by $D^n \hookrightarrow \mathbb{R}^n$, $S^{n-1} \hookrightarrow \mathbb{R}^n - \{x\}$)

incl. (hence since other two arrows are homotopy equivalences, this one is too).



Hence, we have (w/ R-coeffs):

$$
\begin{tikzcd}
H_n(M, M-A) \arrow[r, "\cong"', "\text{incl.}"] \arrow[d, "\cong", "\text{excision}"'] & H_n(M, M-x) \arrow[d, "\cong"', "\text{excision (excise } M\setminus\mathbb{R}^n)"] \\
H_n(\mathbb{R}^n, \mathbb{R}^n - A) \arrow[r, "\text{incl.} \cong"', "\text{by above}"] & H_n(\mathbb{R}^n, \mathbb{R}^n - x) \\
& H_n(D^n, \partial D^n) \arrow[ul, "\cong", "\text{by above}"'] \arrow[u, "\cong"', "\text{by above}"]
\end{tikzcd}
$$

(check homotopy inverse to incl. : $\partial D^n \hookrightarrow \mathbb{R}^n - A$)

$\cong$ $(D^n, \partial D^n)$ 'good pair'

$H_n(D^n/\partial D^n \cong S^n) \cong R$, $\square$.

<u>Shorthand</u>: $H_n(M|x; R) := H_n(M, M-x; R)$ (& leave out $R \Rightarrow$ working over $\mathbb{Z}$ unless otherwise stated).

work over $R = \mathbb{Z}$ for this def'n:

<u>Def</u>: A <u>local orientation</u> of M^n (manifold) at $x \in M$ is a choice of generator $\mu_x \in H_n(M|x) \cong \mathbb{Z}$ (non-canonically).

(two choices of generator, since working over $\mathbb{Z}$).

An <u>orientation on M</u>, if it exists, is a choice of local orientations $\{\mu_x\}_{x \in M}$ which varies 'coherently' or 'continuously' in a suitable sense.

(to be discussed next time)