Last time:

<u>Shorthand</u>: $H_n(M|x;R) := H_n(M, M-x; R)$ (& leave out $R \Rightarrow$ working over $\mathbb{Z}$ unless otherwise stated).

work over $R = \mathbb{Z}$ for this def'n:

<u>Def</u>: A <u>local orientation</u> of M^n (manifold) at $x \in M$ is a choice of generator $\mu_x \in H_n(M|x) \overset{\text{non-canonically}}{\cong} \mathbb{Z}$ (two choices of generator, since working over $\mathbb{Z}$)

<u>Def</u>: An <u>orientation on M</u>, if it exists, is a choice of local orientations $\{\mu_x\}_{x \in M}$ which varies 'coherently' or 'continuously' in a suitable sense.

'<u>Coherent</u>': means that for any $x \in M$, $\exists$ a closed ball $x \ni B \underset{\text{closed ball}}{\hookrightarrow} \mathbb{R}^n \hookrightarrow M$, ~~such~~ such that the induced isomorphism, for any $y \in B$,

means $H_n(M, M-B)$

$$\begin{tikzcd} H_n(M|B) \arrow[r, "\cong", "\text{(by lemma)}"'] \arrow[d, "\cong", "\text{incl. (by lemma)}"'] & H_n(M|x) \arrow[dl, dashed] \\ H_n(M|y) & \end{tikzcd}$$

sends μ_x to μ_y.

continuous? need to define an orientation as a map between top. spaces, from M to ___.

In fact, it will be a (continuous) <u>section</u> of a suitable <u>bundle over M</u> (or $\mathbb{Z}$ (resp. R)-modules); in this case a covering space over M.

Fix R, M^n as above.

Define

$$M_R = \coprod_{x \in M} H_n(M|x;R) = \{\alpha_x \in H_n(M|x;R), x \in M\};$$

means $H_n(M, M-x; R)$

in particular have $M_{\mathbb{Z}}$, $M_{\mathbb{Z}/2}$.

We can topologize M_R by, for any ball $B \subset M$ (w/ say closure $\bar{B}$ giving a closed ball in some $\mathbb{R}^n \hookrightarrow M$), and any $\alpha_B \in H_n(M|\bar{B})$, considering the sets:

$$U(\alpha_B) := \left\{ \alpha_x \in H_n(M|x;R),\ x \;\middle|\; \begin{array}{l} x \in B, \text{ and} \\ \alpha_x = \text{image of } \alpha_B \text{ under} \\ H_n(M|\bar{B}) \xrightarrow{\cong} H_n(M|x) \end{array} \right\}$$

R-coeffs.

$U(\alpha_B) \subset M_R$; these give a basis for the (1st time).

topology we put on M_R.

There is a map $\pi: M_R \longrightarrow M$, which is continuous, and presents M_R as a (infinite-sheeted if e.g. $R=\mathbb{Z}$) covering space
$(\alpha_x, x) \longmapsto x$

over M (we're focusing on the cases $R = \mathbb{Z}$ or $\mathbb{Z}/2$; in general here R is discrete).

(In fact $\pi: M_R \to M$ is a bundle of R-modules over M: every fiber $(M_R)_x := \pi^{-1}(x) = H_n(M|x; R)$ is an R-module, and at every point x $\exists$ $U \ni x$ so that $\pi^{-1}(U) \xrightarrow{\cong} U \times \{$a fixed R-module, in this case $R\}$. (in a way compatible with projections & R-module structures on each fiber)).

Recall a (implicitly continuous) section of a covering space $\tilde{X} \xrightarrow{\pi} X$ is a (continuous) map $s: X \longrightarrow \tilde{X}$ with $s \circ \pi = id_M$.

More generally, a section of a bundle of R-modules is defined the same way; can collect the set of sections of $Y \xrightarrow{\pi} X$

$$\Gamma(Y) := \{s: X \to Y \mid \pi \circ s = id_X\}.$$

in other words $s: x \longmapsto (x, s_x)$

obs: this is an R-module too:
- can add: $(s_1+s_2)(x) := (x, (s_1)_x + (s_2)_x)$
- can mult. by R: $(r \cdot s)(x) := (x, r s_x)$.

Re-def: An orientation (or more generally an R-orientation) of M^n is a section (implicitly continuous)

$$M \xrightarrow{s} M_{\mathbb{Z}} \quad \text{(or more generally } M_R)$$
$$x \longmapsto \mu_x$$

(μ_x shorthand for (μ_x, x))

whose values μ_x at each point generate $H_n(M|x)$ (resp. $H_n(M|x;R)$).

(exercise: compare Redef to original def., i.e., compare 'continuous' to 'coherently varying').

There is a sub covering space $\tilde{M} \subset M_{\mathbb{Z}}$, $\tilde{M} = \{\mu_x \in H_n(M|x) \mid \mu_x \text{ generator}\}$; an orientation in fact gives a section of $\tilde{M}$.

$$\tilde{M} \xrightarrow{\pi} M \xleftarrow{\pi} M_{\mathbb{Z}}$$

(inherits topology from $M_{\mathbb{Z}}$.)

Since we're over $\mathbb{Z}$, each $H_n(M|x)$ has two generators $\Rightarrow$ $\tilde{M}$ is a double cover of M.

We call $\tilde{M}$ the orientation double-cover of M, in light of the above definition & also b/c of:

Lem: $\tilde{M}$ always admits a canonical orientation (even if M doesn't). (note $\tilde{M}$ is a manifold).

Idea: A point $\tilde{x} \in \tilde{M}$ is a pair $\tilde{x} = (\mu_x, x)$ where $\mu_x \in H_n(M|x)$ is a generator.

Observe that $H_n(\tilde{M}|\tilde{x}) \cong H_n(M|x)$; so orient by, at $\tilde{x} = (\mu_x, x)$,
(b/c $\tilde{M}$ covers M sending $\tilde{x}$ to x)

choosing the generator $\mu_x \in H_n(\tilde{M}|\tilde{x}) \cong H_n(M|x)$. $\boxed{\times}$.
exercise: fill in details / check continuous.

On the other hand, M itself may not be orientable (meaning admit an orientation).

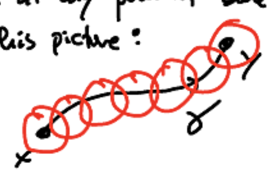
Prop: Say M connected. Then M is orientable $\Longleftrightarrow$ $\tilde{M}$ has two connected components.

Pf: $\tilde{M}$ is a 2-sheeted cover, hence only has 1 or 2 components.

If 2 components: each maps homeomorphically to M, so M is orientable (pick a section by picking one component of $\tilde{M}$ & mapping M to that component by inverse of covering identification)

If M orientable: It has exactly two orientations since it's connected.

(point: given an orientation $\{\mu_x\}_{x\in M}$: μ_x determines μ_y at any point in same component as M by this picture:



So all we can do is swap $\mu_x \mapsto -\mu_x$; this forces $\{\mu_x\}_{x\in M} \rightsquigarrow \{-\mu_x\}_{x\in M}$.).

$\Rightarrow$ $\exists$ exactly two sections $s_1, s_2 : M \longrightarrow \tilde{M}$ ~~dense~~ w/ disjoint images.

Each gives a component of $\tilde{M}$ (point is that given any section $M \xrightarrow{s} \tilde{M}$ of a covering space, $s(M)$ is an entire component of $\tilde{M}$ — why? (exercise: show open + closed)). $\boxed{\times}$.

R-case: A generator in $H_n(M|x;R) \cong R$ is a unit/invertible element.
(sometimes more than 2 elts, sometimes fewer! e.g., $R = \mathbb{Z}/2$)

Note: $H_n(M|x;R) \cong H_n(M|x;\mathbb{Z}) \otimes_{\mathbb{Z}} R$ (by UCT for homology — why? b/c $H_{n-1}(M|x;\mathbb{Z})$ is zero $(n>1)$ or free $(n=1)$).

so each $r \in R$ determines a subcovering space M_r of M_R consisting of all elements of the form $\pm\mu_x \otimes r \in H_n(M|x;R)$, μ_x any generator in $H_n(M|x)$.

- If r is a 2-torsion element (including the case $r=0$), then $r=-r$, so M_r is a copy of M. (i.e., $M_r \xrightarrow{\cong} M$).

- Otherwise $M_r \cong \tilde{M} \cong M_{-r}$, and $M_R = \coprod_{\{r,-r\} \in R/\{\pm 1\}} M_r$.

Using this decomposition, we see that:

(1) An orientable manifold is R-orientable for all R.

(2) A non-orientable manifold is still R-orientable if R contains a unit of order 2. (e.g., if $2=0$ in R).

In particular, every manifold is $\mathbb{Z}/2$-orientable (point: there's always a section of $M_{1\in\mathbb{Z}/2} \subset M_{\mathbb{Z}/2}$ b/c $M_{1\in\mathbb{Z}/2} \cong M$).

Most important cases: $R = \mathbb{Z}, \mathbb{Z}/2$.

Next time: structure theorem for H_n(R-orientable m'flds; R).