Last time: defined notion of R-orientable (meaning $\exists$ R-orientation) $\forall R$, for a manifold M^n. (connected)

- R-orientation $\Longleftrightarrow$ section of $M_R \to M$, $M_R = \coprod_x H_n(M|x;R)$, (w/ topology) s.t., fibers of this section μ_x generate $H_n(M|x;R)$ $\forall x$.
- ($\mathbb{Z}$-)orientable $\Longleftrightarrow$ orientation double cover $\tilde{M} \xrightarrow{2:1} M$ has 2 components.
- any M is $\mathbb{Z}/2$-orientable.

More generally:

- An orientable manifold is R-orientable for all R.
- A non-orientable manifold is still R-orientable if R contains a unit of order 2. (e.g., if $2=0$ in R).

<u>Main theorem</u> (about orientations): M^n connected manifold, R as before

(a) If M is compact and R-orientable, then $H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$ is an isomorphism for every $x \in M$.

can think of as $H_n(M|M;R)$

(b) If M is compact & non-R-orientable, then $H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$ is injective with image $2\text{-Tors}(R) = \{r \in R \mid 2r = 0\}$ for all $x \in M$

(c) If M is non-compact, then $H_n(M;R) = 0$.

(d) $H_i(M;R) = 0$ for $i > n$.

<u>In particular</u>:

- For a cpct connected manifol M^n, $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ or 0 depending on whether M is orientable.

(<u>ex</u>: $H_2(\mathbb{RP}^2;\mathbb{Z}) \underset{\text{by CW homology}}{=} 0$ so $\mathbb{RP}^2$ not orientable. $H_3(\mathbb{RP}^3;\mathbb{Z}) = \mathbb{Z}$ so $\mathbb{RP}^3$ is orientable).

• either way if M cpct, connected, $H_n(M;\mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}$.

Def'n: M orientable and compact. An element of $H_n(M;R)$ whose image in $H_n(M|x;R)$ generates for all x is called a fundamental class for M with R-coeffs, denoted $[M]$. (note this is a choice).

A fund. class $[M] \in H_n(M;\mathbb{Z})$ is a generator, and is equivalent, for a cpct. manifold, to a choice of orientation (as we'll see).

Cor: A fund. class $[M]$ w/ R-coeffs. exists iff M is cpct. and R-orientable.

($\Leftarrow$ Thm, $\Rightarrow$ Say $[M]$ is a fund. class; since $[M] \neq 0$, M cpct, let μ_x be its image in $H_n(M|x;R)$.
observe (exercise): $x \longmapsto (\mu_x, x)$ is an orientation of M).
i.e., is continuous.

More technical statement (than theorem), implies main theorem:

M^n connected manifold, (not. nec. cpct.) A closed subset of M, and given $M_R \xrightarrow{\pi} M$, consider $(M_R)|_A := \pi^{-1}(A)$; have $(M_R)|_A \xrightarrow{\pi|_A} A$, & denote its sections by $\Gamma(A;(M_R)|_A)$.

Lem: (a) There is a bijection, for A compact:

$$\Gamma(A;(M_R)|_A) \xleftarrow[J_A]{1:1} H_n(M|A;R)$$

$$s_{\alpha_A} \longleftarrow\!\shortmid \alpha_A$$

defined by

$s_{\alpha_A} : x \longmapsto (\alpha_A)|_x$ denotes image of α_A under $H_n(M|A;R) \longrightarrow H_n(M|x;R)$

(For A not necessarily compact — we won't prove this case —

$$(*) \qquad \Gamma_c(A;(M_R)|_A) \xleftarrow[J_A]{\cong} H_n(M|A;R) \quad)$$

sections w/ cpct. support, meaning (defined as above).

$s = 0$ for x outside compact

$s_x = 0$ for x outside a compact set in base.

(b) $H_i(M|A;R) = 0$ for $i > n$, A closed.

Claim: Lem $\Rightarrow$ Main Theorem. Assume lemma, for all A.

- part (b) implies $(A = M)$ $H_i(M;R) = 0$ $i > n$.

- if M is non-compact, observe that

$$\Gamma_c(M; M_R) = 0,$$ because a section of a covering space (if it exists) is determined on any connected component by what it does at a single point (covering space theory)

(M cpct, set $A = M$)

- It suffices by lemma to study restr.

$$H_n(M;R) \cong \Gamma(M; M_R) \xrightarrow{\text{restr.}} (M_R)_x := H_n(M|x;R) \cong R \quad \text{for any } x \in M.$$
$$s \longmapsto s_x.$$

* restr is always injective, b/c any section of it exists is determined by its values at a point (M is connected), by covering space theory.

- If M is orientable, and $r \in R$ then there is a section of M_R taking value r at $(M_R)_x$, b/c $M_r \cong \begin{cases} \widetilde{M} \\ M \end{cases}$ and we can find a section of M over both $\widetilde{M}$ and M, hence over $M_r \subseteq M_R$.

- If M is not orientable, then we can only find a section of M_r when r is 2-torsion, hence the image of restr consists of 2-torsion. $\boxtimes$.

Pf of technical lemma: (sketch, in the case A is compact).

omit R from notation for this proof, for simplicity.

The idea is to induct on the size of A and M.

Let $\underline{P_M(A)}$ be the statement that $J_A : H_n(M|A) \xrightarrow{\cong} \Gamma(A; (M_R)|_A)$ is an iso.

Claim 1*: If $P_M(A)$, $P_M(B)$, and $P_M(A \cap B)$ hold, then $P_M(A \cup B)$ holds.

Assume Claim 1 for now. Then, we can already reduce to the case of $M=\mathbb{R}^n$, A some cpct. set.

How? If $A \subseteq M$ cpct subset, can write $A = A_1 \cup \dots \cup A_m$ where each A_i is cpct and contained in an open $\mathbb{R}^n \subset M$. (why? exercise).

Note first of all that if $A_i \subset \mathbb{R}^n \subset M$, then $P_M(A_i) \Longleftrightarrow P_{\mathbb{R}^n}(A_i)$
(b/c by excision $H_n(M|A_i) \cong H_n(\mathbb{R}^n|A_i)$).

Assuming $P_{\mathbb{R}^n}(B)$ holds for any B cpct for a moment, suppose inductively that $P_M(A_1 \cup \dots \cup A_{m-1})$ and $P_M(A_m)$, The intersection $(A_1 \cap A_m) \cup \dots \cup (A_{m-1} \cap A_m)$ is

for any cpct $A_i \subset$ some $\mathbb{R}^n \subset M$.

again a union of $(m-1)$ cpct subsets of Euclidean charts, so $P_M((A_1 \cup \dots \cup A_{m-1}) \cap A_m)$ holds too.

Then Claim 1 $\Rightarrow$ $P_M(A_1 \cup \dots \cup A_m)$ holds.

Next, note: for $M=\mathbb{R}^n$, A=convex subset, then $P_{\mathbb{R}^n}(A)$ holds because we've already shown

$$\begin{array}{ccc} H_n(M|A) & \xrightarrow{\cong} & H_n(M|x) \quad (x \in A) \\ \downarrow J_A & & \mathrm{id}=J_x \downarrow \cong \\ \Gamma(A, M_R) & \xrightarrow[\cong]{\text{restr.}} & \Gamma(x; M_R) \end{array} \qquad \Rightarrow J_A \cong.$$

(by contractibility.)
exercise

What to do for an arbitrary compact set $A \subset \mathbb{R}^n$? If $A = \bigcup_{\text{finite}}$ convex sets, we're done by Claim 1.
(b/c intersection of convex sets is convex).

Idea is that any A can be 'approximated' by unions of convex sets, in the sense that $\exists$ $E_1, E_2, E_3, \dots$, seq. of compact sets in $\mathbb{R}^n$

with $E_1 \supset E_2 \supset E_3 \supset \cdots$, each E_i is a (finite) union of convex sets,

and $\bigcap E_i = A$.

(e.g., pick $\delta_1, \delta_2, \delta_3, \dots$, $\delta_i \to 0$, let E_1 be any finite cover of A by closed δ_1-balls,

and let E_k be intersection of E_{k-1} w/ any finite cover of A by δ_k-balls

(intersections preserve the property of being a finite union of convex sets)).

How does this help?

Claim 3*: If $P_M(A_i)$ holds for $A_1 \supset A_2 \supset A_3 \supset \cdots$ seq. of cpct. subsets then $P_M(A = \bigcap A_i)$ holds.

(in light of above, it follows $P_{\mathbb{R}^n}(A)$ holds for any cpct. A hence $P_M(A)$ holds for any A).

Assuming Claim 2, we learn that since an arbitrary cpct $A \subseteq \mathbb{R}^n$ is $\bigcap_i E_i$ where each E_i is a union of convex sets — $P_M(A)$ holds for an arbitrary cpct. A.

Thus, modulo Claims 1 & 2, this proves the technical Lemma, hence the Main theorem

∎