Direct limits of direct systems of R-modules (or objects in a category $\mathcal{C}$)

Let $(S, \leq)$ directed set, meaning a set S w/ partial order $\leq$ such that for any $\alpha, \beta \in S$ $\exists\, \gamma$ with $\alpha \leq \gamma, \beta \leq \gamma$.

(can think of any poset $(S, \leq)$ as a category w/ objects $=$ ob S & $\hom(a,b) = \begin{cases} * & a \leq b \\ \phi & \text{else} \end{cases}$, the directed condition $\Rightarrow$ $(S, \leq)$ is a filtered category, a def'n we won't elaborate on here)

(Ex: (1) $(\mathbb{N}, \leq)$, (2) $(\{\text{open subsets of a top. space } X\}, \subseteq)$).

A direct system (of objects in $\mathcal{C}$) indexed by S is a functor $(S, \leq) \to \mathcal{C}$;

in other words e.g., it consists of the following data:

$\{G_\alpha, \alpha \in S\}$ and $\psi_{\alpha\beta}: G_\alpha \to G_\beta$ when $\alpha \leq \beta$ (R-module maps if $\mathcal{C} = R$-Mod)

R-modules if $\mathcal{C} = R$-Mod

with $\psi_{\beta\gamma}\psi_{\alpha\beta} = \psi_{\alpha\gamma}$ if $\alpha \leq \beta \leq \gamma$,

(E.g., if $(S, \leq) = (\mathbb{N}, \leq)$, this is equivalent to a sequence $G_1 \xrightarrow{\psi_{12}} G_2 \xrightarrow{\psi_{23}} G_3 \to \cdots$ (seeing as $\psi_{k\ell} = \psi_{\ell-1,\ell} \circ \psi_{\ell-2,\ell-1} \circ \cdots \circ \psi_{k,k+1}$.))

The direct limit of such a direct system, denoted $\varinjlim_{\alpha \in S} G_\alpha$ is defined to be (if it exists)

any object of $\mathcal{C}$ (R-module) $\overline{G}$, equipped with maps $G_\alpha \xrightarrow{f_\alpha} \overline{G}$ for every $\alpha \in S$ s.t.

the following diagram commutes $\forall\ \alpha \leq \beta \in S$

$$\begin{tikzcd} G_\alpha \arrow[rr, "\psi_{\alpha\beta}"] \arrow[dr, "f_\alpha"'] & \circlearrowleft & G_\beta \arrow[dl, "f_\beta"] \\ & \overline{G} & \end{tikzcd},$$

which satisfy the following universal property: for any other H equipped with maps $G_\alpha \xrightarrow{g_\alpha} H$

s.t. $\begin{tikzcd} G_\alpha \arrow[rr, "\psi_{\alpha\beta}"] \arrow[dr, "g_\alpha"'] & \circlearrowleft & G_\beta \arrow[dl, "g_\beta"] \\ & H & \end{tikzcd}$ $\forall\ \alpha \leq \beta$, there exists a unique map $\overline{G} \xrightarrow{g} H$ factoring g_α as $g \circ f_\alpha$:

$$\begin{tikzcd} G_\alpha \arrow[rr, "\psi_{\alpha\beta}"] \arrow[dr, "f_\alpha"] \arrow[ddr, bend right, "g_\alpha"'] & & G_\beta \arrow[dl, "f_\beta"'] \arrow[ddl, bend left, "g_\beta"] \\ & \overline{G} \arrow[d, dashed, "\exists !"] & \\ & H & \end{tikzcd}$$

Lemma: If $\varinjlim_{\alpha\in S} G_\alpha$ exists, it's unique up to unique isomorphism.

(i.e., any two $\overline{G}$, $\overline{G}'$ satisfying universal property are uniquely isomorphic).

(Exercise using univ. property).

Lemma: $\mathcal{C} = R\text{-Mod}$. Then $\varinjlim_{\alpha\in S} G_\alpha$ exists for any direct system $(S, \le) \to R\text{-Mod}$.

Pf sketch: Given a direct system of R-Modules $\{G_\alpha\}_{\alpha\in S}$, $\{\psi_{\alpha\beta}: G_\alpha \to G_\beta\}_{\alpha\le\beta\in S}$,

with $\psi_{\beta\gamma}\circ\psi_{\alpha\beta} = \psi_{\alpha\gamma}\ \forall\, \alpha\le\beta\le\gamma$,

explicitly define

$$\varinjlim G_\alpha := \overline{G} := \{(g,\alpha) \mid g\in G_\alpha,\ \alpha\in S\} \Big/ (g,\alpha)\sim(h,\beta)$$

if $\exists\ \gamma$ with $\alpha\le\gamma$, $\beta\le\gamma$,

and $\psi_{\alpha\gamma}(g) = \psi_{\beta\gamma}(h)$ in G_γ, .

and $f_\alpha : G_\alpha \longrightarrow \overline{G}$

$(g,\alpha) \longmapsto [(g,\alpha)]$

Exercise: Well-defined, has a natural well defined R-module structure given by

$r\cdot[(g,\alpha)] := [(rg,\alpha)]$, satisfies universal property, along w/

$\exists\ G_\alpha \xrightarrow{f_\alpha} \overline{G}$

$g \longmapsto [(\alpha, g)]$

$\Rightarrow$ existence. $\boxtimes$.

In particular, for any sequence of R-modules $M_1 \to M_2 \to \text{—}$

$\varinjlim_i M_i$ exists & is characterized by its universal property.

(equivalently a natural transformation of functors $(S,\le) \overset{G}{\underset{H}{\rightrightarrows}} \mathcal{C}$)

Lemma: Given a morphism of direct systems over $(S,\le)$, meaning $G_\alpha \to H_\alpha\ \forall\,\alpha$ s.t.

$$\begin{tikzcd} G_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G_\beta \arrow[d] \\ H_\alpha \arrow[r] & H_\beta \end{tikzcd}$$

$\forall\ \alpha \leq \beta$, there is an induced map $\varinjlim_{\alpha\in S} G_\alpha \xrightarrow{(*)} \varinjlim_{\alpha \in S} H_\alpha$ (when these exist, e.g., for R-modules)

if all $G_\alpha \to H_\alpha$ are isomorphisms, then $\varinjlim G_\alpha \xrightarrow{\cong} \varinjlim H_\alpha$ too. (Exercise).

From last time: M^n connected manifold, $A \subseteq M$ closed subsets cpct., $R :=$ coeffs. (implicit).

$H_n(M|A) := H_n(M, M-A)$, $\quad \begin{matrix} M_R \\ \downarrow \pi \\ M \end{matrix} :=$ bundle of R-modules w/ fibers $(M_R)_x := \pi^{-1}(x) = H_n(M|x)$.

$M_R|_A := \pi^{-1}(A) \xrightarrow{\pi} A$. all fibers lying are points of A.

$\Gamma(A;(M_R)|_A) :=$ R-module of sections over A.

$\mathrm{rest}_x : H_n(M|A) \to H_n(M|x)$ for any $x \in A$.
(comes from $M \to M$, $M-A \hookrightarrow M - \{x\}$)

J_A: map $H_n(M|A) \longrightarrow \Gamma(A; (M_R)|_A)$
$\alpha_A \longmapsto s_{\alpha_A} : x \longmapsto \mathrm{rest}_x(\alpha_A)$.

Let $P_M(A)$ be the statement that

$$J_A : H_n(M|A) \xrightarrow{\cong} \Gamma(A;(M_R)|_A) \text{ is an iso.}$$

Then modulo two claims we'd proven a technical lemma about this being an iso. $\forall$ M, A cpt $\subseteq M$, & a theorem about $H_n(M^n;R)$ & $H_{>n}(M^n;R)$ (depending on R-orientability & compactness of M)

Claim 1*: If $P_M(A)$, $P_M(B)$, and $P_M(A\cap B)$ hold, then $P_M(A \cup B)$ holds.

Claim 2*: If $P_M(A_i)$ holds for $A_1 \supset A_2 \supset A_3 \supset \cdots$ seq. of cpct. subsets then $P_M(A = \bigcap A_i)$ holds.

We'll start with Claim 2, which makes use of the theory of direct limits.

Observe that for any $B \subseteq A \subseteq M$, have $(M, M-A) \hookrightarrow (M, M-B)$
inducing $H_n(M|A) \to H_n(M|B)$ s.t., if $C \subseteq B \subseteq A$,
the following diagram commutes:

$$\begin{tikzcd} H_n(M|A) \arrow[rr] \arrow[dr] & \circlearrowleft & H_n(M|C) \\ & H_n(M|B) \arrow[ur] & \end{tikzcd}$$

in the setting of Claim 2,

$\Rightarrow$ have a sequence (hence direct system) of R-modules

$$\begin{tikzcd}
H_n(M|A_1) \arrow[r] \arrow[dr] & H_n(M|A_2) \arrow[r] \arrow[d] & \cdots \arrow[dl] \\
& H_n(M|A) &
\end{tikzcd}$$

along with maps

where ($A = \bigcap A_i$)

<u>Lemma</u>: The induced map $\varinjlim_i H_n(M|A_i) \xrightarrow{\cong} H_n(M|A)$ is an iso.

<u>Pf</u>: mostly omitted, but we want to indicate one key idea

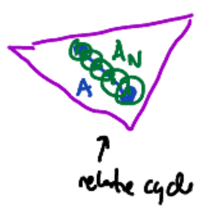
why is $\varinjlim_i H_p(M|A_i) \longrightarrow H_p(M|A)$ surjective?

$$\begin{array}{ccc}
\varinjlim_i H_p(M|A_i) & \longrightarrow & H_p(M|A) \\
\shortparallel & & \shortparallel \\
\varinjlim_i H_p(M, M-A_i) & \longrightarrow & H_p(M, M-A) \\
& & \Downarrow \\
& & [\sigma]
\end{array}$$

The first observation is that any relative cycle σ in $(M, M-A)$ has $\partial\sigma$ <u>compact</u>, hence supported in a compact subset of $M-A$. This implies its disjoint from some A_N, $N>>0$, hence contained in $(M, M-A_N)$.

(why? <u>exercise</u>:

idea: (in $\mathbb{R}^n$)



$\partial\sigma$ and A have a minimum distance δ, hence $\partial\sigma$ doesn't touch any cover of A by closed $\delta/2$ balls either).

∎

Similarly, given that for any $B \subseteq A$ we have a natural restriction of sections map $\Gamma(A; (M_R)|_A) \longrightarrow \Gamma(B|(M_R)|_B)$ s.t., for

$$s \longmapsto s|_B$$

$C \subseteq B \subseteq A$, restriction from A to C = rest. from A to B, then from B to C.

$\Rightarrow$ a sequence (direct system)

$$\begin{tikzcd}
\Gamma(A_1;(M_R)|_{A_1}) \arrow[r] \arrow[dr] & \Gamma(A_2;(M_R)|_{A_2}) \arrow[r] \arrow[d] & \cdots \\
& \Gamma(A;(M_R)|_A) & (A = \cap A_i)
\end{tikzcd}$$

and maps

Lemma: The induced map $\varinjlim_i \Gamma(A_i;(M_R)|_{A_i}) \xrightarrow{\cong} \Gamma(A;(M_R)|_A)$ is an iso.

(Exercise)

Pf of Claim 2: (sketch): Check there's a morphism of sequences

$$\begin{tikzcd}
H_n(M|A_1) \arrow[r] \arrow[d, "J_{A_1}", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & H_n(M|A_2) \arrow[r] \arrow[d, "J_{A_2}", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \cdots \\
\Gamma(A_1;(M_R)|_{A_1}) \arrow[r] & \Gamma(A_2;(M_R)|_{A_2}) \arrow[r] & \cdots
\end{tikzcd}$$

iso. by hypotheses

i.e., check diagram commutes

$$\begin{tikzcd}
\varinjlim_i H_n(M|A_i) \arrow[r] \arrow[d, "\varinjlim J_{A_i}"] \arrow[dr, phantom, "\circlearrowright \text{ check}"] & H_n(M|A) \arrow[d] \\
\varinjlim_i \Gamma(A_i;(M_R)|_{A_i}) \arrow[r] & \Gamma(A;M_R|
\end{tikzcd}$$

Proof of Claim 1: Suppose $P_M(A)$, $P_M(B)$, & $P_M(A\cap B)$ hold.

Idea: First observe for G abelian, $H_1, H_2 < G$, there's a SES

$(g_1+H_1, g_2+H_2) \longmapsto (g_1 - g_2 + H_1 + H_2)$.

$$0 \to \frac{G}{H_1\cap H_2} \to \frac{G}{H_1}\oplus\frac{G}{H_2} \to \frac{G}{H_1+H_2} \to 0,$$

$g+(H_1\cap H_2) \longmapsto (g+H_1, g+H_2)$

For $V_1, V_2 \underset{\text{open}}{\subset} X$ recall we defined "$C_\bullet(V_1+V_2)$" := sum $\underset{\text{in}}{\overset{\text{"}H_1\text{"}}{C_\bullet(V_1)}} + \overset{\text{"}H_2\text{"}}{C_\bullet(V_2)}$ $C_\bullet(X) \leftarrow$ "G"

$\Rightarrow$ a SES $\quad 0 \to C_\bullet(X, V_1\cap V_2) \to C_\bullet(X,V_1)\oplus C_\bullet(X,V_2) \to C_\bullet(X, V_1+V_2) \to 0$

know $H_\bullet$ of this computes $H_\bullet(X, V_1\cup V_2)$ from befor

$\boxed{\text{LES}}$ $\Rightarrow$

("M-V upside down")

$$\cdots \to H_{n+1}(X, V_1\cup V_2) \to H_n(X, V_1\cap V_2) \to H_n(X,V_1)\oplus H_n(X,V_2) \to H_n(X, V_1\cap V_2) \to \cdots$$

our case: study the case
- $X = M$
- $V_1 = M - A$
- $V_2 = M - B$
- $V_1\cap V_2 = M - (A\cup B)$
- $V_1\cup V_2 = M - (A\cap B)$.

$H_i(M,V_1), H_i(M,V_2)$

$H_i(M, V_1\cup V_2) = 0$, when $i > n$ by assumpn. So M-V implies immediately that

$H_i(M, V_1\cap V_2) = 0$ for $i > n$ too (sandwiched between 0's in LES).

We also get a diagram of SES's:

$H_{n+1}(M, V_1\cup V_2)$
$\|$

$$\begin{tikzcd}
0 \arrow[r] & H_n(M|A\cup B) \arrow[r, "(\mathrm{restr}_A,\,\mathrm{restr}_B)"] \arrow[d, "J_{A\cup B}"] & H_n(M|A)\oplus H_n(M|B) \arrow[r, "(\text{difference of restrictions})"] \arrow[d, "J_A\oplus J_B", "\wr"'] & H_n(M|A\cap B) \arrow[d, "J_{A\cap B}", "\wr"'] \\
0 \arrow[r] & \Gamma(A\cup B, M_R) \arrow[r, "(\mathrm{restr}_A,\,\mathrm{restr}_B)"'] & \Gamma(A;M_R)\oplus\Gamma(B;M_R) \arrow[r, "(\text{difference of restrns})"'] & \Gamma(A\cap B; M_R)
\end{tikzcd}$$

Exercise: Check lower SES — a mere general fact about sections — & commute diagram.

Therefore, 5-Lemma $\Rightarrow$ $J_{A\cup B}$ is an isomorphism as desired. $\boxtimes$.