Poincaré duality:

{ Right now $R = \mathbb{Z}$
{ implicitly, can work w/ R-coeffs, then M R-orientable; i.e., M always $\mathbb{Z}/2$-orientable

Thm says (first version):

dimension (M).

If M^n orientable, cpct manifold, then $H^\ell(M) \cong H_{n-\ell}(M)$

(M R-orientable, $H^\ell(M;R) \xrightarrow{\cong} H_{n-\ell}(M;R)$).

The isomorphism is given by cap product with a fundamental class:

recall that we have cap product action $H_n(X) \times H^\ell(X) \longrightarrow H_{n-\ell}(X)$,

assume M connected

and if M R-orientable, a fund. class is a choice of generator $[M] \in H_n(M;R) \cong R$

$\Longleftrightarrow$ a choice of section of $M_R \to M$ which generates at each fiber, i.e., an R-orientation.

$\rightsquigarrow D_M := [M] \frown (-) : H^\ell(M;R) \longrightarrow H_{n-\ell}(M;R)$

duality isomorphism.

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originally historically phrased in terms of existence of a dual polyhedral subdivision to a given sufficiently fine triangulation.

point $\longleftrightarrow$ top-dim'l face

1-simplex $\longleftrightarrow$ codim-1 face.

compatible in a dual sense w/ boundary operators.

$\vdots \qquad \vdots$

Corollaries of Poincaré duality

M oriented, n'dim'l, cpct.

knew this

In principle could have had $\mathrm{Ext}(H_{n-1},\mathbb{Z})$ contributions.

(1) If M connected, then $H_n(M) = \mathbb{Z}$, and $H^n(M) = \mathbb{Z}$ (b/c $H^0(M) = \mathbb{Z}$ and $H_0(M) = \mathbb{Z}$).

(2) Let's use the notation

$$\overline{H} := H/\mathrm{Tors}(H), \text{ for a } \mathbb{Z}\text{-module } H.$$

Poincaré duality implies there's a perfect pairing on $\overline{H^\bullet}(X)$ resp. $\overline{H_\bullet}(X)$.

(Recall if $\Gamma_1 \cong \mathbb{Z}^n$, $\Gamma_2 \cong \mathbb{Z}^n$, a bilinear $q: \Gamma_1 \times \Gamma_2 \to \mathbb{Z}$ is perfect if

$$q^*: \Gamma_1 \xrightarrow{\cong} \mathrm{Hom}(\Gamma_2, \mathbb{Z}) \iff \text{for any } \mathbb{Z}\text{-bases of } \Gamma_1, \Gamma_2, \text{ matrix of } q$$
$$e \longmapsto q(e,-) \qquad \text{has } \det \pm 1.$$
(unimodular)

To spell out the details, let's recall first that

<u>Thm</u>: M cpct mfold. Then $H_\ell(M)$ is a finitely generated $\mathbb{Z}$-module for all ℓ.
(we'll omit details, see Hatcher).

appeals to classification of f.g. $\mathbb{Z}$-mods

Using this, we learn $H_\ell(M) = \mathbb{Z}^r \oplus \text{Torsion}$, & $\mathrm{Ext}(H_{\ell-1}(M), \mathbb{Z}) \cong \mathrm{Tors}(H_{\ell-1}(M))$.

UCT tells us that $H^\ell(M) \to \mathrm{Hom}(H_\ell(M), \mathbb{Z})$ is surjective w/ kernel the torsion of $H_{\ell-1}(M)$
$$[\phi] \longmapsto \beta([\phi])([\sigma]) := \phi(\sigma).$$
$\quad = $ torsion of $H^\ell(M)$

$$\Rightarrow \text{get } \overline{H^\ell}(M) \xrightarrow[\beta]{\cong} \mathrm{Hom}(\overline{H_\ell}(M), \mathbb{Z}).$$

means mod torsion

by this fact.

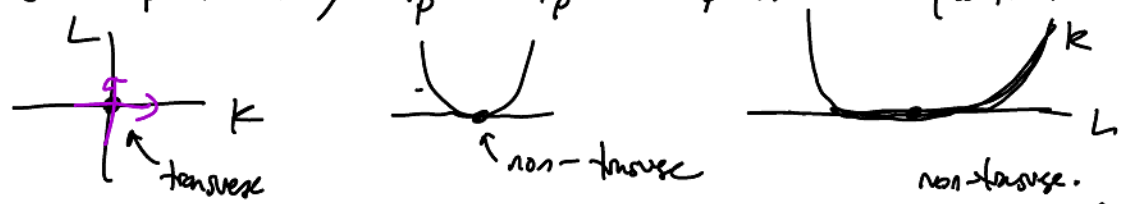
$\cong \mathrm{Hom}(H_\ell(M), \mathbb{Z})$
(b/c $\mathrm{Hom}(H, \mathbb{Z})$ kills $\mathrm{tors}(H)$).

i.e., Have a perfect pairing $\overline{H^\ell(M)} \times \overline{H_\ell(M)} \to \mathbb{Z}$
$$\langle [\phi], [\sigma] \rangle = \phi(\sigma).$$

P.D. $\Rightarrow \exists$ a perfect pairing
$$\overline{H_{n-\ell}(M)} \times \overline{H_\ell(M)} \to \mathbb{Z}.$$
$$\overline{H^\ell}(M)$$
$$(\gamma_1, \gamma_2) \longmapsto \gamma_1 \cdot \gamma_2 := \langle D_M^{-1}\gamma_1, \gamma_2 \rangle$$
"intersection pairing" (why?)

Geometrically, if $K^{n-\ell}, L^\ell \subseteq M^n$ compact oriented submanifolds of M (cpct oriented)

Let's assume further K, L, M smooth, and K, L <u>intersect transversely</u>, meaning at each $p \in K \cap L$, $T_pK + T_pL = T_pM$. (write $K \pitchfork L$)



(points carry signs: do the sums of orientations on K, L match orientation on M at p?)

when K, L transverse, $K \cap L$ is a cpct <u>oriented</u> 0-manifold. (= finite union of points),

$$\Rightarrow K \cdot L := \sum_{p \in K \cap L} \mathrm{sign}(p)$$

geom. intersection #

± 1 depending on

an isotopy of a submanifold $K \hookrightarrow M$

is a smooth homotopy i_t, with each i_t an embedding.

If $K \not\pitchfork L$, we can isotope it to be $\pitchfork$ & then intersect.

Intersection # is an isotopy invariant so result is invariant.

defined using P.D.

Thm (omitted here): For K, L as above, $K \cdot_{geom} L = [K] \cdot [L] \in$ in $H_\ell(M)$.

means look at image $[K]$ in $H_{n-\ell}(K) \to H_{n-\ell}(M)$.

Duality in terms of cup product.

Thm: (coh. intersection pairing) M^n cpct, (R-)oriented, R explicit. Then, the pairing

$$\begin{array}{l} \overline{H}^p(M) \otimes \overline{H}^{n-p}(M) \xrightarrow{\cup} \overline{H}^n(M) \\ \qquad\qquad\qquad\qquad\qquad \downarrow \cong \ (-)\cap[M] \ (\text{or } D_M) \\ \qquad\qquad\qquad\qquad\qquad \overline{H}_0(M) \\ \qquad\qquad\qquad\qquad\qquad \downarrow \cong \\ \qquad\qquad\qquad\qquad\qquad R. \end{array}$$

is a perfect pairing.

(mod torsion)

sometimes called "$\int_M (-)$."

> Recall: if $[\alpha] \in H^\ell(X)$, $[\beta] \in H_\ell(X)$, then $\alpha(\beta) := \varepsilon_*([\alpha] \cap [\beta])$, where $[\alpha] \cap [\beta] \in H_0(X)$, and $\varepsilon_* : H_0(X) \xrightarrow{\cong} R$ (for X connected)

Pf (from P.D.)

$[\phi] \mapsto \{[6] \mapsto \phi(6)\}$

Have $\overline{H}^p(X) \xrightarrow[\text{UCT}]{\cong} \text{Hom}(\overline{H}_p(X), R) \xrightarrow[D_M^*]{\cong} \text{Hom}(\overline{H}^{n-p}(X), R)$ $(-\circ D_M)$.

This map is given by

$$[\phi] \longmapsto \{[\psi] \mapsto \phi(\overbrace{[\psi] \cap [M]}^{D_M([\psi])})\}$$

$$= \varepsilon_*([\phi] \cap ([\psi] \cap [M]))$$

module property

$$= \varepsilon_*(([\phi] \cup [\psi]) \cap [M])$$

$$= (\phi \cup \psi)([M]). \ \blacksquare$$

chain level version of fund. class

Application: coh. rings of projective spaces

Prop: $H^*(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1} \quad |\alpha| = 1$

$H^\bullet(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}$ $\quad |\alpha|=2$ $\qquad$ as rings.

$H^\bullet(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1}$ $\quad |\alpha|=4$

<u>Pf:</u> Let's do $\mathbb{CP}^n$ (other proofs are the same). Induction on n:

$n=1$: $H^\bullet(\mathbb{CP}^1;\mathbb{Z}) \cong H^\bullet(S^2;\mathbb{Z}) \underset{\text{already know}}{\cong} \mathbb{Z}[\alpha]/\alpha^2$ $\quad |\alpha|=2$. ✓.

<u>Inductive step</u>: assume true for $\mathbb{CP}^{n-1}$. $\quad (n>1)$

$\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a $2n$ cell, so

LES of $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ in cohomology $\Rightarrow$ the restriction

$r^*: H^i(\mathbb{CP}^n) \xrightarrow{\cong} H^i(\mathbb{CP}^{n-1})$ for $i \le 2n-2$. $\qquad$ (where $r: \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$)

(why? <u>exercise</u>)

By naturality of cup product, we learn that if $\alpha \in H^2(\mathbb{CP}^n)$ generator, then $r^*\alpha$ generates $H^2(\mathbb{CP}^{n-1})$,

$$\Rightarrow \underbrace{(r^*\alpha)^i}_{\substack{\| \text{ naturality} \\ r^*(\alpha^i)}} \text{ generates } H^{2i}(\mathbb{CP}^{n-1}) \quad i \le n-1 \quad \text{(by induct. step)}.$$

$\Rightarrow \alpha^i$ generates $H^i(\mathbb{CP}^n)$ $\quad i \le 2n-2$.

So have elements $\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$ generating $H^2, H^4, \ldots, H^{2n-2}$.

<u>Q:</u> is $\underbrace{\alpha \cup \alpha^{n-1}}_{\alpha^n}$ a generator of $H^{2n}(\mathbb{CP}^n)$? (if so, we're done)

Yes, by Poincaré duality: $\mathbb{CP}^n$ is a cpct (connected) $2n$ manifold, & $H_{2n}(\mathbb{CP}^n) \cong \mathbb{Z}$, so orientable. So $\exists$ perfect pairing (choosing $[\mathbb{CP}^n]$):

$$H^2(\mathbb{CP}^n) \otimes H^{2n-2}(\mathbb{CP}^n) \xrightarrow{\ \cup\ } H^{2n}(\mathbb{CP}^n) \xrightarrow{D_M} H_0(\mathbb{CP}^n) \overset{\cong}{\to} \mathbb{Z}.$$

($\Rightarrow$ a generator $\cup$ a generator must be a generator.). $\qquad$ ∎.

Idea in proof of P.D.:

Again by induction/covering argument, want to reduce to case of $\mathbb{R}^n$.

The local case $\mathbb{R}^n$ is a non-compact manifold, for which duality as stated fails $\left(\begin{array}{l} \text{e.g.,} \\ H_n(\mathbb{R}^n) = 0 \\ n > 0 \end{array} \right)$.

We need a formulation of P.D. which holds in non-compact setting too, which is suitably functorial allowing for induction. We'll get this by replacing $H^l \rightsquigarrow H^l_c$ "compactly supported cohomology".