Relative cap product

Recall had a cap product operation $C^\bullet(X) \otimes C_\bullet(X) \xrightarrow{\frown} C_\bullet(X)$.

Observations/exercises:

- If $A \subset X$, and c is a chain in A, then $\alpha \frown c$ is a chain in A too, for any $\alpha \in C^\bullet(X)$, so get $\frown : C^p(X) \otimes C_n(X,A) \longrightarrow C_{n-p}(X,A)$
  (chain map)
  $\Rightarrow H^p(X) \otimes H_n(X,A) \xrightarrow{\frown} H_{n-p}(X,A)$

- Also get: $\underset{\text{Ann}(A)}{\underset{\shortparallel}{C^p(X,A)}} \otimes C_n(X,A) \xrightarrow{\frown} C_{n-p}(X)$

  $\text{Ann}(C_\bullet(A)) \otimes C_\bullet(X) \xrightarrow{\frown} C_\bullet(X)$ sends $\text{Ann}(C_\bullet(A)) \otimes C_\bullet(A) \to 0$, hence get the desired induced map $\text{Ann}(C_\bullet(A)) \otimes \frac{C_\bullet(X)}{C_\bullet(A)} \to C_\bullet(X)$.

  idea: elements here are are zero on $C_\bullet(A) \subset C_\bullet(X)$, so unaffected by adding chains chains on A.

  $\Rightarrow H^p(X,A) \otimes H_n(X,A) \xrightarrow{\frown} H_{n-p}(X)$

- More generally, if $A, B \underset{\text{open}}{\subset} X$, $\left(C_\bullet(\text{“}A+B\text{”}) \simeq C_\bullet(A \cup B) \right)$,
  means $C_\bullet(A) + C_\bullet(B)$ (in $C_\bullet(X)$).

  get $C^p(X,A) \otimes C_n(X, \text{“}A+B\text{”}) \xrightarrow{\frown} C_{n-p}(X,B)$, inducing
  meaning $\frac{C_n(X)}{C_n(A)+C_n(B)}$ (recall, homology of this is $H_n(X, A\cup B)$ by excision)

  $\Rightarrow H^p(X,A) \otimes H_n(X, A\cup B) \to H_{n-p}(X,B)$.

Local formulation of Poincaré Duality (for not nec. cpct manifolds).

Say M non-compact mfld, $K \subset M$ cpct. subset. R coeffs. (suppressed).

Recall from above the cap product for $(M, M-K)$:

$$H_n(M, M-K) \times H^\ell(M, M-K) \longrightarrow H_{n-\ell}(M).$$

$$\exists!\, \mu_K \in H_n(M|K).$$
$\quad J_K \big\downarrow \cong$ technical lemma (K cpct.)

$\Gamma(K; M_R)$ $\ni s|_K$ orientations

- If M (R-)orientable, pick orientation ($s: M \to M_R$ whose image generates $(M_R)_x$ at every x).
- Restrict to K, $s|_K \in \Gamma(K; M_R)$.
- Technical lemma say $\exists!\, \mu_K \in H_n(M|K)$ restricting to $s|_K$. Call it the 'local fundamental class' (requires K cpct.). (note if M non-compact $H_n(M) = 0$).

Naively might hope that

$$- \cap \mu_K : H^\ell(M, M-K) \longrightarrow H_{n-\ell}(M) \text{ is an iso. for all } M,\ K \underset{\text{cpct}}{\subset} M.$$

(if true, would imply P.D. when M cpct b/c set $K = M$)

This is not exactly true, but it ends up being true in a limiting sense as we let K get arbitrarily large.

(It cannot be, as e.g., for $M = \mathbb{R}^n$, $K = \coprod_{i=1}^{\ell}$ points, cohomology depends on $\# K$)

<u>Note</u>: If $K_1 \subseteq K_2$ cpct. sets, then

$$(M, M-K_2) \xrightarrow[i_{K_2,K_1}]{\text{incl.}} (M, M-K_1), \text{ and the element } \overbrace{\mu_{K_2} \text{ maps to } \mu_{K_1}}^{\star} \text{ (check).}$$

Also get $i^*_{K_2,K_1} : H^\ell(M, M-K_2) \to H^\ell(M, M-K_1)$

and if $K_1 \subseteq K_2 \subseteq K_3$ then $i^*_{K_2K_1} \circ i^*_{K_3K_2} = i^*_{K_3K_1}$.

so if we let $S = \{K \mid K \subset M \text{ cpct. subsets}\}$, ordered by $\subseteq$, note S is a <u>directed set</u>

and $\{H^*(M, M-K)\}_{K \in S}$, $\{i^*_{L,K}\}_{K \subseteq L}$ is a <u>direct system</u> <u>of R-modules</u> indexed by S.

<u>Def</u>: The <u>compactly supported cohomology</u> of a (not. nec. compact) manifold M is:

$$H^\ell_c(M) := \varinjlim_{\substack{K \subset M \text{ cpct.} \\ (\text{i.e. } K \in S)}} H^\ell(M, M-K)$$

(explicitly, H^ℓ_c is given by co-chains $\varphi \in C^*(M)$ w/ $\varphi \equiv 0$ on all chains in $M-K$ for some $K \subseteq M$).

For $K_1 \subseteq K_2$ we claim the following diagram commutes by naturality of cap product (using $\star$) with respect to i_{K_2,K_1} :

$$\begin{tikzcd}
H^\ell(M, M-K_1) \arrow[d, "i^*_{K_2,K_1}"'] \arrow[dr, "\frown \mu_{K_1}"] \arrow[d, phantom, shift left=8, "\circlearrowleft"] & \\
& H_{n-\ell}(M) \\
H^\ell(M, M-K_2) \arrow[ur, "\frown \mu_{K_2}"'] &
\end{tikzcd}$$

This implies, by univ. property of direct limit, there is a unique induced map

$$D_M := \varinjlim_{\substack{K \subset M \\ \text{cpct.}}} (- \frown \mu_K) \quad : \quad H^\ell_c(M) \longrightarrow H_{n-\ell}(M).$$

Rmk: If M cpct, then $S =. \{K \subset M \text{ cpct.}\}$ contains a maximal element, M itself.

By definition of direct limit, can directly verify that if S has a maximal element $G_{\max}$ then

$$G_{G_{\max}} \xrightarrow{\cong} \varinjlim_{\alpha \in S} G_\alpha .$$

In this case, we see that

$$\begin{tikzcd}
H^\ell(M,M-M) \arrow[d, phantom, "\shortparallel"] & \\
H^\ell(M) \arrow[r, "\cong"] \arrow[dr, "D_M"'] & H^\ell_c(M) \arrow[d, "D_M"] \\
& H_{n-\ell}(M).
\end{tikzcd}$$

Thm: (Poincaré duality for non-compact manifolds):

If M is oriented, then

$$D_M := \varinjlim_{\substack{K \subset M \\ \text{cpct.}}} (- \frown \mu_K) \quad : \quad H^\ell_c(M) \xrightarrow{\cong} H_{n-\ell}(M).$$

induced by choice of orientation.

(Rmk above says this recovers P.D. for compact manifolds)

Idea of proof:

Induct on M. Let $P(M)$ be the statement above for a given M.

Claim 1: True when $M = \mathbb{R}^n$ (hence true when M = ball in $\mathbb{R}^n$)

Claim 2: If $M = U \cup V$, U, V open, & $P(U)$, $P(V)$, $P(U \cap V)$ hold, then

$P(U \cup V) = P(M)$ holds.

(w/ step 1 $\Rightarrow$ true for any finite union of open balls in $\mathbb{R}^n$).

Claim 3: (limits): If $P(-)$ holds for each of $U_1 \underset{\text{open}}{\subset} U_2 \underset{\text{open}}{\subset} U_3 \subset \cdots$ (all open in some M)
then $P(\bigcup U_i)$ holds.

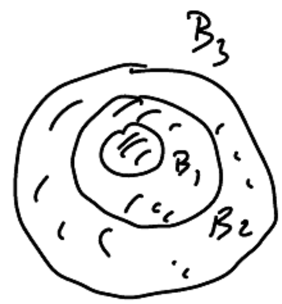
$\Rightarrow$ $P(-)$ holds for any open in $\mathbb{R}^n$ (can always express any $U \overset{\text{open}}{\subset} \mathbb{R}^n$ as union of countably many open balls, & let U_k be union of first k balls. By Step 2 $P(U_k)$ holds & $U_1 \subset U_2 \subset \cdots \underset{\text{step 3}}{\Rightarrow} P(U := \bigcup U_i)$ holds)

$\Rightarrow$ $P(-)$ holds for any finite (by Step 2) & the countable (by Step 3) union of open sets in M which are homeomorphic to $\mathbb{R}^n$.

(note given $U_1 \subset M$, $U_1 \cong \mathbb{R}^n$, $U_2 \subseteq M$, $U_2 \cong \mathbb{R}^n$; $U_1 \cap U_2 \cong$ an open in $\mathbb{R}^n$. need step 3 to get $P(U_1 \cap U_2)$, then $P(U_1 \cup U_2)$ by step 2).

$\Rightarrow$ $P(-)$ holds for M. (we'll assume M has a countable base for simplicity only.)

Pf of Claim 1:



Idea: $\mathbb{R}^n$ is exhausted by cpct. subsets $\overline{B_i(0)}$, $i \in \mathbb{N}$.

$\Rightarrow$ $\overline{B_i(0)}$ is cofinal in $\{K \subset \mathbb{R}^n \text{ compact subsets}\}$

($T \subseteq S$ is cofinal in S if every $s \in S$ is $\leq$ some $t \in T$)
(direct systems, direct system $\Rightarrow$ $\varinjlim_{s \in S} G_s \xleftarrow{\cong} \varinjlim_{t \in T} G_t$).

$$\Rightarrow H^l_c(\mathbb{R}^n) \cong \varinjlim_i H^l(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B_i(0)})$$

general properties of direct limit

$\cong$ excision

$H^l(D^n, D^n \setminus \overline{B_i(0)})$ D^n very large disk (maybe depends on i)

$\cong$ htpy inv.

$H^l(D^n, S^{n-1})$

$\cong$ good pair.

$$\tilde{H}^l(D^n/S^{n-1}) \cong \tilde{H}^l(S^n) \cong \begin{cases} \mathbb{Z} & l = n \\ 0 & \text{otherwise.} \end{cases}$$

Study:

$H^\ell(\mathbb{R}^n; \mathbb{R}^n \setminus B_i(0)) \longrightarrow H_{n-\ell}(\mathbb{R}^n)$.

- $\cong$ when $\ell \neq n$ b/c both sides are 0.
- when $\ell = n$, $\cong$ b/c $\mu_{\overline{B_i(0)}}$ is a generator of $H_n(\mathbb{R}^n | \overline{B_i(0)})$,

  and UCT says that

$$\begin{array}{ccc} H^n(\mathbb{R}^n; \mathbb{R}^n \setminus B_i(0)) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}}(H_n(\mathbb{R}^n | \overline{B_i(0)}), \mathbb{Z}) \cong \mathbb{Z} \\ [\phi] & \longmapsto & \phi(\overline{\mu_{B_i(0)}}) = \varepsilon_*([\phi] \frown [\overline{\mu_{B_i(0)}}]). \end{array}$$

Hence $- \frown [\mu_{B_i(0)}]$ is an isomorphism.

Now, provided we know that

$$\begin{tikzcd} H^\ell(\mathbb{R}^n, \mathbb{R}^n \setminus B_i(0)) \arrow[r, "\cong"] \arrow[dr, "- \frown {[\mu_{B_i(0)}]}"', "\cong"] & H^\ell(\mathbb{R}^n; \mathbb{R}^n \setminus B_j(0)) \arrow[d, "- \frown {[\mu_{B_j(0)}]}", "\cong"'] \\ & H_{n-\ell}(\mathbb{R}^n) \end{tikzcd}$$

for $j > i$, we're done. $\boxed{\varepsilon}$.

(exercise)