Thm: (Poincaré duality for non-compact manifolds):

If M is oriented, then

$$D_M := \varinjlim_{\substack{K\subset M\\ \text{cpct.}}} (- \frown \mu_K) \; : \; H^\ell_c(M) \xrightarrow{\cong} H_{n-\ell}(M).$$

induced by choice of orientation.

Last time, reduced thm inductively to 3 claims:

(where $P(M)$ be the statement above for a given M.)

Claim 1: True when $M = \mathbb{R}^n$ (hence true when M = ball in $\mathbb{R}^n$)

Claim 2: If $M = U \cup V$, U, V open, & $P(U)$, $P(V)$, $P(U\cap V)$ hold, then $P(U \cup V) = P(M)$ holds.

(w/ step 1 $\Rightarrow$ true for any finite union of open balls in $\mathbb{R}^n$).

Claim 3: (limits): If $P(-)$ holds for each of $U_1 \underset{\text{open}}{\subset} U_2 \underset{\text{open}}{\subset} U_3 \subset \cdots$ (all open in some M) then $P(\bigcup U_i)$ holds.

We also proved Claim 1.

Today: sketch Claims 2, 3.

A flavor of how Claim 2 is proved:

The key claim is if $M = U \cup V$, U, V open in M,

Lem: $\exists$ a commutative diagram of Mayer-Vietoris LES:

a covariantly induced pushforward for $U \subset M$! $V \subset M$!

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}", "(1)"'] & H^k_c(U) \oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V", "(2)"'] & H^k_c(M) \arrow[r, "\delta^*"] \arrow[d, "D_M", "(3)"'] \arrow[dr, phantom, "\circlearrowleft \text{ is a bit technical}"] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}", "(4)"'] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r, "\partial_*"] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

usual M-V LES

(Pf: Hatcher p. 246 Lemma 3.36).

in homology.

Assuming lem, if $P(U)$, $P(V)$, $P(U\cap V)$ hold, then (1), (2), (4) are $\cong$,
hence 5-lemma $\Rightarrow$ (3) $\cong$, so $P(M)$ holds.

One observation is that H^ℓ_c is in fact covariantly functorial for open inclusions $U_1 \underset{\text{open}}{\subset} U_2$.

i.e., $U_1 \underset{\text{open}}{\overset{i}{\subset}} U_2 \rightsquigarrow i_! : H^\ell_c(U_1) \to H^\ell_c(U_2)$ "extension by zero".

These maps appear in top LES, (w.r.t. $U\cap V \hookrightarrow U$, $U \cap V \hookrightarrow V$, $U, V \hookrightarrow M$.

A Flavor of Claim 3:

The main idea is that $\left(U_1 \underset{\text{open}}{\overset{i_1}{\subset}} U_2 \underset{\text{open}}{\overset{i_2}{\subset}} \cdots \right) \subseteq M$

induces

$$\begin{tikzcd}
H^\ell_c(U_1) \arrow[r, "(i_1)_!"] \arrow[dr, "(j_1)_!"'] & H^\ell_c(U_2) \arrow[r, "(i_2)_!"] \arrow[d, "(j_2)_!"] & \cdots \arrow[dl] \\
& H^\ell_c(\bigcup U_i) &
\end{tikzcd}$$

$i_m : U_m \hookrightarrow U_{m+1}$

$j_m : U_m \hookrightarrow \bigcup U_i$

and also

$$\begin{tikzcd}
H_{n-\ell}(U_1) \arrow[r, "(i_1)_*"] \arrow[dr, "(j_1)_*"'] & H_{n-\ell}(U_2) \arrow[r, "(i_2)_*"] \arrow[d, "(j_2)_*"] & \cdots \\
& H_{n-\ell}(\bigcup U_i) &
\end{tikzcd}$$

hence:

$$\begin{tikzcd}
\varinjlim H^\ell_c(U_i) \arrow[r, "\varinjlim (j_m)_!"] \arrow[d, "\varinjlim D_{U_i}"', "\cong\ (\text{given})"] & H^\ell_c(\bigcup U_i) \arrow[d, "D_{\cup U_i}\ (\cong ??)"] \\
\varinjlim H_{n-\ell}(U_i) \arrow[r, "\varinjlim (j_m)_*"] & H_{n-\ell}(\bigcup U_i)
\end{tikzcd}$$

Main Claim: These are both $\cong$.

Exercise: verify main claim. (basic idea for homology is e.g., that any $\sigma : \Delta^n \to \bigcup U_i$ has image in some U_N, $N \gg 0$).

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there are many generalizations of Poincaré duality, we'll focus on one such for manifolds with boundary ("Lefschetz duality" or "Poincaré-Lefschetz", ...)

Def: An n-manifold with boundary is a Hausdorff space M which is locally homeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n = \{x_1 \geq 0\} \subseteq \mathbb{R}^n$.

equality allowed.

Obs:

- If $x \in M$ has a nhood homeo. to $\mathbb{R}^n$, then excision as before implies that $H_n(M|x) \left(\stackrel{\text{def}}{=} H_n(M, M-x) \right) \cong \mathbb{Z}$.

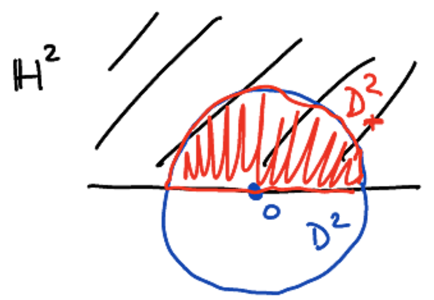
(WLOG $x=0$)

- If $x \in M$ has a nhood homeo. to $\mathbb{H}^n$ in a way sending x to a point with $x_1 = 0$, then excision $\Rightarrow H_n(M|x) \stackrel{\text{excision}}{\cong} H_n(\mathbb{H}^n, \mathbb{H}^n - \{0\})$

$$\cong \text{ excision}$$

$$= H_n(\underbrace{D^n \cap \mathbb{H}^n}_{D^n_+}, D^n_+ \setminus 0)$$

note: both convex sets!
(in contrast, $D^n \setminus 0$ not convex but $D^n_+ \setminus 0$ is)



$$= 0.$$

We conclude that if x is sent to a boundary point in one $\mathbb{H}^n$ local model, it must be sent to a boundary point in every $\mathbb{H}^n$ local model.

The boundary points of M, denoted ∂M, are those x with $H_n(M|x) = 0$.

(<=> those x around which $\exists$ an identification w/ $\mathbb{H}^n$ sending x to the boundary of $\mathbb{H}^n$).

e.g., $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$!

& more generally, $\partial M = (n-1)$-manifold.

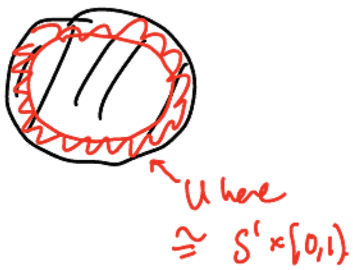
examples:

$\partial D^n = S^{n-1}$,

$\partial M = S^1$.

A <u>collar neighborhood</u> of ∂M in M is a nhood U of ∂M (in M) homeomorphic to $\partial M \times [0,1)$ (in a way identifying $\partial M \times \{0\} \underset{\text{id}}{\cong} \partial M$).

<u>Prop</u>: Any compact manifold with boundary has a collar neighborhood around ∂M.

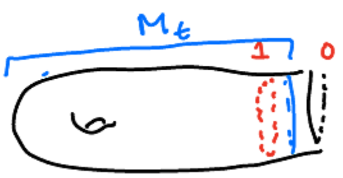


We'll omit the proof, but note in the smooth case it can be proven by flowing for small time by an <u>inward-pointing vector field</u>

(in turn, an inward-pointing vec. field exists by a partition of unity argument.

General case: Hatcher's book.

<u>Useful consequences of having a collar nhood</u>:



M.

Fix $U \underset{\text{open}}{\supset} \partial M$ with $U \underset{\text{homeo}}{\cong} \partial M \times [0,1)_r$

We'll define for any $t \in (0,1)$, $M_t := M \setminus \left(\begin{array}{l}\text{image in } M \text{ of } \{r<t\} \\ \text{in } U\end{array}\right)$.

<u>Observe</u>: $M_t \xhookrightarrow{\text{incl.}} M$ is a homotopy equivalence, and moreover is homotopic to a homeomorphism which is the identity outside the collar, and in the collar, is any homeo

$$[t,1) \xrightarrow{\cong} [0,1) \quad \text{which is the identity near } 1.$$

(gives $\partial M \times [t,1) \cong \partial M \times [0,1)$, now extend by identity to get $M_t \xrightarrow{\cong} M$).

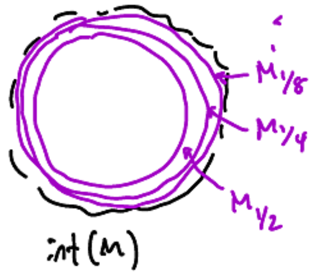
More generally, for $t_1 < t_2$, $M_{t_2} \xhookrightarrow[\text{incl.}]{} M_{t_1}$ is a homotopy equivalence.)

Now look at $\text{int}(M) := M \setminus \partial M$. This is a not necessarily compact manifold (compact iff M cpct & $\partial M = \emptyset$). The above choice of collar nhood & M_t's give us an <u>exhaustion</u> of $\text{int}(M)$ by compact sets

$$M_{s_1} \subset M_{s_2} \subset M_{s_3} \subset \cdots$$

where $s_1 > s_2 > s_3 > \cdots$ is a sequence tending to zero.

e.g.,



Also, each $M_{s_1} \hookrightarrow \mathrm{int}(M)$ is a homotopy equivalence.

Orientations on manifolds with boundary:

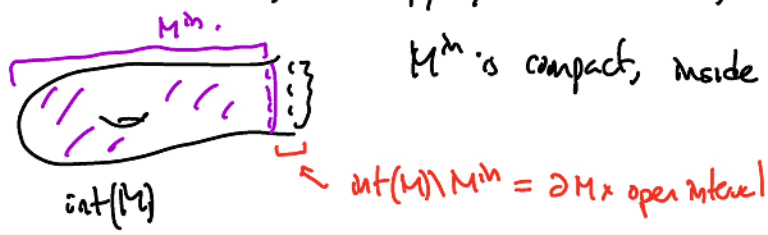
M manifold with boundary. Say M is orientable if $\mathrm{int}(M)$ is orientable, & an orientation on M means an orientation on $\mathrm{int}(M)$.

As before, we can define $\begin{matrix} M_R \\ \downarrow \\ \mathrm{int}(M) \end{matrix}$ covering space ('bundle of R-modules') whose fiber at $x \in \mathrm{int}(M)$ is $H_n(M|x; R)$.

Sections of M_R which generate at each point $\longleftrightarrow$ R-orientations

Pick M^{in} m'fold with boundary inside $\mathrm{int}(M)$ homotopy equivalent to $\mathrm{int}(M)$,

i.e., $M^{in} := M_{1/2}$.



M^{in} is compact, inside $\mathrm{int}(M)$, so

Technical lemma w/ $K = M^{in} \hookrightarrow \mathrm{int}(M)$ implies:

$$\begin{array}{ccccc}
\Gamma(M^{in}; M_R) & \cong & H_n(\mathrm{int}(M) \,|\, M^{in}) & = & H_n(\underbrace{\mathrm{int}(M)}, \underbrace{\mathrm{int}(M)\setminus M^{in}}_{\partial M \times \text{interval}}) \\
\| 2 & & \uparrow \| 2 & & \\
\Gamma(\mathrm{int}(M); M_R) & & H_n(M_{1/4}, \partial M_{1/4}) & & \\
 & & \| 2 & & \\
 & & H_n(M, \partial M). & &
\end{array}$$

$M^{in} \simeq \mathrm{int}(M)$ h.e.

htpy equiv. to $M_{1/4}$

$\| 2$ h.e., $\partial M \times \{\tfrac{1}{4}\}$.

Cor: (over $\mathbb{Z}$, similar over other R's) M (cpct connected m'fold w/ ∂) is ($\mathbb{Z}$-) orientable iff $H_n(M, \partial M) = \mathbb{Z}$.

A fund. class (choice of generator in $H_n(M, \partial M)$) $\Longleftrightarrow$ a choice of orientation.

Thm: (Poincaré duality for manifolds with boundary). M^n cpct with boundary, orientable, fix $[M] \in H_n(M, \partial M)$ (R-coeffs / R-orientations implicit). ($\Longleftrightarrow$ choice of R-orientation on M).

$\Rightarrow$ get maps which are isomorphisms

(1) $D_M = (-) \cap [M] : H^\ell(M, \partial M) \xrightarrow{\cong} H_{n-\ell}(M)$

(2) $D_M = (-) \cap [M] : H^\ell(M) \xrightarrow{\cong} H_{n-\ell}(M, \partial M)$.