Thm: (Poincaré duality for manifolds with boundary). M^n cpct with boundary, orientable, fix $[M] \in H_n(M, \partial M)$ (R-coeffs / R-orientations implicit). ($\Leftrightarrow$ choice of R-orientation on M).

$\Rightarrow$ get maps which are isomorphisms

(1) $\quad D_M = (-) \cap [M] : H^\ell(M, \partial M) \xrightarrow{\cong} H_{n-\ell}(M)$

(2) $\quad D_M = (-) \cap [M] : H^\ell(M) \xrightarrow{\cong} H_{n-\ell}(M, \partial M)$.

The first observation is that (2) follows from (1) and

Lemma: $\exists$ comm. diagram of duality maps between LES's associated to the pair $(M, \partial M)$:
(exercise)

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(M,\partial M) \arrow[r] \arrow[d, "D_M\,(1)", "\text{by assumption}\ \cong"'] & H^k(M) \arrow[r] \arrow[d, "D_M\,(2)"] & H^k(\partial M) \arrow[r, "\delta^*"] \arrow[d, "D_{\partial M}\ \cong\ \text{b/c } \partial M \text{ is compact,}"] & H^{k+1}(M,\partial M) \arrow[r] \arrow[d, "D_M\,(1)\ \cong\ \text{by assumption}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k}(M,\partial M) \arrow[r, "\partial_*"] & H_{n-k-1}(\partial M) \arrow[r] & H_{n-k-1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

(part of this lemma: an orientation on M induces one on ∂M, compatible w/

$$\partial_* : H_n(M, \partial M) \to H_{n-1}(\partial M)$$
$$[M] \longmapsto \text{a choice of fund. class in } H_{n-1}(\partial M) \Big).$$

So 5-lemma + (1) $\Rightarrow$ (2).

How to see Thm (1) from non-compact Poincaré duality?

Non-compact P.D. implies:

$$H^\ell_c(\mathring{M}) \xrightarrow[\cong]{D_M} H_{n-\ell}(\mathring{M})$$

Now we'll use the fact that $\exists$ an exhaustion of $\mathring{M}$ by compact sets

$$(M_1 \subset M_2 \subset M_3 \subset \cdots) \text{ in } \mathring{M},$$ (using a collar nhood)

with each $M_i \subset M_{i+1}$ a homotopy equiv, & each $M_i \underset{\text{homeo.}}{\cong} M$. (uses collar nhood).

In particular $\{M_i\}$ is cofinal in $(\{\text{cpct } K \subset \mathring{M}\}, \subseteq)$

so $$H^\ell_c(\mathring{M}) \cong \varinjlim_i H^\ell(\mathring{M}, \mathring{M}\setminus M_i)$$

$$\uparrow \cong \text{ excision}$$

$$\varinjlim_i H^\ell(\mathring{M}_{big}, \mathring{M}_{big}\setminus M_i)$$

$$\downarrow \cong$$

b/c $\mathring{M} \supseteq M_i$ for each i. and $\mathring{M}_{big}\setminus M_i \xhookrightarrow{\simeq} \mathring{M}_{big}\setminus \mathring{M}$.

$$H^\ell(\mathring{M}_{big}, \mathring{M}_{big}\setminus \mathring{M}) \underset{\text{excision}}{\cong} H^\ell(M, \partial M).$$

($M \xhookrightarrow{\simeq} \mathring{M}_{big}$, $\partial M \xhookrightarrow{\simeq} \mathring{M}_{big}\setminus\mathring{M}$).

$M_{big} = M \cup_{\partial M} \partial M \times [-\varepsilon, 0]$ identified using collar.
(i.e., glue $\partial M\times[-\varepsilon,\varepsilon]$ to $\partial M\times[0,1)$ along $\partial M\times[0,\varepsilon]$).

(i.e., for a manifold-with-boundary, $\boxed{H^\ell_c(\mathrm{int}(M)) \cong H^\ell(M, \partial M)}$).

& moreover, want to check:

$$\begin{tikzcd}
H^\ell_c(\mathring{M}) \arrow[r, "\cong D_M"] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-\ell}(\mathring{M}) \arrow[d, phantom, "\cong"] \\
H^\ell(M, \partial M) \arrow[r, "D_M", "(1)"'] & H_{n-\ell}(M)
\end{tikzcd}$$

(exercise).

$\Rightarrow$ (1) is an isomorphism. ∎.

New topic: fiber bundles, vector bundles, principal bundles

special examples of fiber bundles, with more structure.

Def: A fiber bundle over B is a space E w/ a map $\pi : E \longrightarrow B$ (continuous), satisfying

the 'fiber' of E at b

(local triviality): for every $b \in B$, denoting $E_b := \pi^{-1}(b)$, $\exists$ open $U \ni p$ in B and a map $E|_U := \pi^{-1}(U) \xrightarrow{t} E_b$ such that the map

$$E|_U \xrightarrow{(\pi, t) = \varphi} U \times E_b$$

is a homeomorphism. (note φ fits into a comm. diagram

$$\begin{tikzcd} E|_U \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & & U \times E_b \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

)

proj. to first factor.

Note: any two fibers of a fiber bundle in the same connected component of B must be homeomorphic. We'll often just restrict to a connected B or assume all fibers homeo.

Example: (1) F any space, form

$$\begin{array}{c} X \times F \\ \downarrow \pi_X \\ X \end{array}$$

trivial fiber bundle w/ fiber F.

(2) covering space

$$\begin{array}{c} \tilde{X} \\ \downarrow \pi \\ X \end{array}$$

is a fiber bundle w/ discrete fibers.

(3) (non-discrete, non-trivial example):

$S^3 \subset \mathbb{C}^2$ unit sphere, & consider $\pi : S^3 \longrightarrow \mathbb{CP}^1 = S^2$

'Hopf fibration'

$v \longmapsto \{\text{complex line in } \mathbb{C}^2 \text{ through } 0 \text{ \& } v\}$

(concretely, $\underbrace{S^3 \hookrightarrow \mathbb{C}^2 \setminus 0 \xrightarrow{\text{quotient}} \mathbb{CP}^1}_{\pi}$)

This gives a fiber bundle over S^2 whose fibers are all (S^1)'s. (b/c $\mathrm{span}_{\mathbb{C}}(\underset{S^3}{v}) = \mathrm{span}_{\mathbb{C}}(\underbrace{e^{i\theta} v}_{S^3})$).

This is not a trivial fiber bundle (i.e. not isomorphic to one): $S^3 \neq S^2 \times S^1$ (e.g., H_i's are different)

(4) $V_k(\mathbb{R}^n)$ Stiefel manifold

exercise from 535a: show this is a smooth manifold.

$= \{\text{orthogonal } k\text{-frames in } \mathbb{R}^n\} = \{A \in \mathrm{Mat}(n \times k) \mid AA^T = \mathrm{Id}_k\}.$

This is a compact manifold. (How to see this? To start, observe $O(n)$ acts on $V_k(\mathbb{R}^n)$ by composition: transitive action, & isotropy group of basepoint $\{e_1, \dots, e_k\}$ is $I_k \times O(n-k)$. $\underset{\text{using this, can show}}{\Longrightarrow}$ $V_k(\mathbb{R}^n) = O(n) / I_k \times O(n-k)$

compact

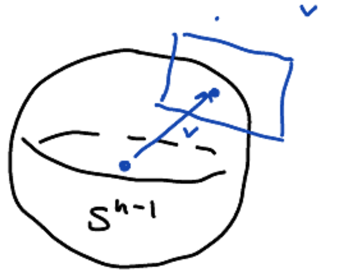
($\Rightarrow$ Hausdorff, cpct).

- Get a fiber bundle (fibration) $O(n) \to V_k(\mathbb{R}^n)$ with fiber $O(n-k)$. (why? locally trivial?) e.g., why

  e.g., $V_1(\mathbb{R}^n) = S^{n-1}$, so in particular $O(n) \to S^{n-1}$ w/ fiber $O(n-1)$.

- Forget last $(k-1)$ vectors: $V_k(\mathbb{R}^n) \to V_1(\mathbb{R}^n) = S^{n-1}$. with fiber at $v \in S^{n-1}$ the collection of $(k-1)$ tuples of orthogonal frames that are orthogonal to v, i.e., $(k-1)$-orthogonal frames of $T_v S^{n-1}$.

tangent space = vectors $\perp$ v



The basic results that allow for us to show examples in (*) are fiber bundles (& many other examples) are:

Thm: (Ehresmann): Say E, B smooth manifolds, $\pi: E \to B$. smooth map. If π is

- proper (i.e., $\pi^{-1}(\text{cpct.})$ is cpct)
- submersion (means $d\pi_x: T_x E \to T_x B$ surjective for all x).

then $\pi: E \to B$ is a fiber bundle.

Using this, can prove:

Prop: G Lie group, and $K \subseteq H \subseteq G$ closed subgroups (so K, H also Lie groups) then the projection map

$$G/K \longrightarrow G/H$$
$$g+K \longmapsto g+H.$$

is a fiber bundle with fibers isomorphic to H/K.

can apply this general result to get examples in (*), and many others.