Last time:

- defined notion of a fiber bundle.
- Prop: G Lie group, and $K \subseteq H \subseteq G$ closed subgroups (so K, H also Lie groups) then the projection map

$$G/K \longrightarrow G/H$$
$$g+K \longmapsto g+H.$$

is a fiber bundle with fibers isomorphic to H/K.

- $\Rightarrow O(n) \longrightarrow V_k(\mathbb{R}^n) = O(n)/I_k \times O(n-k)$ fiber bundle w/ fibers $O(k)$

Another example: Grassmannians.

Def: $G_k(\mathbb{R}^n)$ (or $Gr_{\mathbb{R}}(k,n)$) $:= \{V \subset \mathbb{R}^n \mid V \text{ a real linear } k\text{-dim'l subspace}\}$.

$G_1(\mathbb{R}^{n+1}) := \mathbb{RP}^n$.

(There's also a complex version:

$G_k(\mathbb{C}^n) := \{V \subset \mathbb{C}^n \mid V \text{ a cplx-linear } k\text{-dim'l subspace}\}$.

w/ $G_1(\mathbb{C}^{n+1}) := \mathbb{CP}^n$., w/ same construction.)

can explicitly construct as

$$G_k(\mathbb{R}^n) = \{\text{linearly independent } k\text{-tuples in } \mathbb{R}^n\}/GL(k,\mathbb{R})$$

open subset of $(\mathbb{R}^n)^k$.

applying $GL(k,\mathbb{R})$ to a tuple gives same span.

equipped w/ quotient topology,

can also construct as

$k \times n$ matrices A w/ $AA^T = Id_k$.

$$= \{\text{orthonormal } k\text{-tuples in } \mathbb{R}^n\}/O(k)$$

$$= \{\text{orthonormal } n\text{-tuples in } \mathbb{R}^n\}/O(k)\times O(n-k)$$

$$= O(n)/O(k)\times O(n-k)$$

can check again that $G_k(\mathbb{R}^n)$ is a cpct, Hausdorff manifold.

The Prop above implies: $V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ is a fiber bundle w/ fibers $O(k)$.

$\{v_1, \dots, v_k\} \longmapsto \operatorname{span}(v_1, \dots, v_k)$

As we'll see, many of the above examples have the structure of principal bundles.

Vector bundles

a type of fiber bundle where all fibers are vector spaces (& trivializations are compat. w/ this structure).

X a space.

Def: A real vector bundle over X is

(i) a space E

(ii) a continuous $\pi : E \to X$

(iii) a real vector space structure on each $E_x := \pi^{-1}(x)$, $x \in X$.

Satisfying (local triviality):

for every $x_0 \in X$, $\exists$ a nhood $U \ni x_0$ in X and a homeo, for some n

$$\begin{tikzcd} E|_U = \pi^{-1}(U) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times \mathbb{R}^n \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

π_U (proj. to first factor)

s.t. $\varphi|_{E_x} : E_x \xrightarrow[(\text{by } \circlearrowleft)]{} \{x\} \times \mathbb{R}^n \cong \mathbb{R}^n$ is a real linear isomorphism, for each $x \in U$.

Similarly, have notion of a complex vector bundle: (replace real by complex & $\mathbb{R}^n$ by $\mathbb{C}^n$).

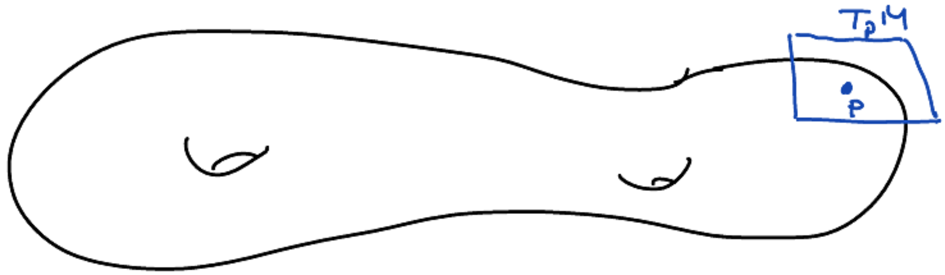
Examples:

(i) $X \times \mathbb{R}^n =: \underline{\mathbb{R}}^n$ equipped w/ $\underset{\pi_X}{\pi} : X \times \mathbb{R}^n \longrightarrow X$ (projection to X)

trivial vector bundle.

(ii) M any smooth (C^∞) manifold, then its tangent bundle $TM \xrightarrow{\pi} M$ (fiber at $p \in M$ is T_pM tangent space) is a vector bundle.

e.g., if $M \subset \mathbb{R}^N$



(so are T^*M, $\Lambda^k T^*M$, etc.)

(iii) M^m smooth manifold, $N^n \subseteq M^m$ smooth submanifold.

Then $\exists$ a vector bundle $\nu_N M$, the normal bundle to $N \subset M$ w/ fiber at $p \in N$ equal to T_pM / T_pN.

(construction is a special case of the fact that can take (pointwise) quotient of a vec. bdle ($TM|_N$) by a sub-bundle ($TN \subset TM|_N$), and result is again a vec. bdle ($TM|_N / TN$).).

(iv) Tautological vector bundles on Grassmannians

Define $E_{taut} \xrightarrow{\pi} Gr_k(\mathbb{R}^n)$ by:

(similarly $E_{taut} \longrightarrow Gr_k(\mathbb{C}^n)$ tautological complex vec. bundle)

$$E_{taut} \subseteq Gr_k(\mathbb{R}^n) \times \mathbb{R}^n$$
$$:= \{(X, v) \mid X \in Gr_k(\mathbb{R}^n),\ v \in X\}$$

and $\pi(X, v) := X$.

the point

the subspace of $\mathbb{R}^n$

Observe: $(E_{taut})_X := \pi^{-1}(X) = \{X\} \times X \cong X$, i.e., has a linear structure.

Local triviality?

Choose a surjection $\alpha : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^k$ (linear).

true whenever $X \cap \overbrace{\ker(\alpha)}^{(n-k)\text{-dim'l}} = \{0\}$.

Define $U_\alpha := \{X \in Gr_k(\mathbb{R}^n) \mid \alpha|_X : X \longrightarrow \mathbb{R}^k \text{ is an isomorphism}\}$

(open dense subset, and $\{U_\alpha\}_{\alpha \in Surj(\mathbb{R}^n, \mathbb{R}^k)}$ cover $Gr_k(\mathbb{R}^n)$)

On U_α have a trivialization

$$E|_{U_\alpha} \xrightarrow{\varphi_\alpha} U_\alpha \times \mathbb{R}^k$$

check (exercise):

- homeomorphism, compat. w/ projections.
- linear in each fiber.

$(x, v) \longmapsto (x, \alpha|_x(v))$.

$(x \in U_\alpha, v \in X)$

<u>Def</u>: The <u>rank</u> of $E \xrightarrow{\pi} X$ is $\dim_{\mathbb{R} \text{ or } \mathbb{C}}(E_x)$, provided this number is constant in X.
(over $\mathbb{R}$ or $\mathbb{C}$) — any x

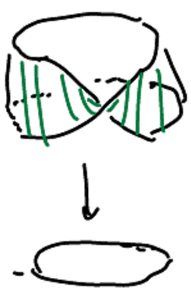
(know it has to be locally constant b/c local triviality, we'll usually assume global constancy so we can talk about rank)

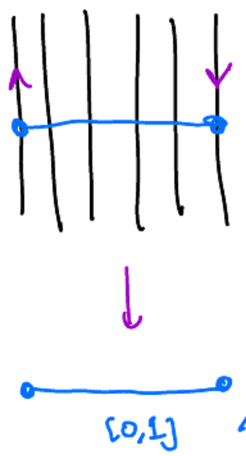
(real or complex) <u>line bundle</u>: vector bundle of (real or complex) rank $= 1$.

e.g., tautological bundles over $Gr_k(\mathbb{R}^n)$ $Gr_k(\mathbb{C}^n)$ when $k = 1$ give:

- $L_{taut} \longrightarrow \mathbb{C}P^n$ tautological (complex) line bundle
- $L_{taut} \longrightarrow \mathbb{R}P^n$ tautological (real) line bundle

<u>subexample/exercise</u>: Look at $L_{taut} \longrightarrow \mathbb{R}P^1 \cong S^1$ & verify L_{taut} is $\cong$ <u>Möbius bundle</u>:





$[0,1] \times \mathbb{R} \,/\, (0,v) \sim (1,-v)$

$[0,1] \,/\, 0 \sim 1$.

& verify L_{taut} is <u>not trivial</u>.

<u>Def</u>: An <u>isomorphism of vector bundles</u> $E \xrightarrow{\pi_E} X$, $F \xrightarrow{\pi_F} X$ is a homeomorphism,

compat. w/ projections:
$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & \arrow[d, phantom, "\circlearrowleft"] & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd}$$
, such that $\varphi|_{E_x}: E_x \to F_x$ is a linear isomorphism for each $x \in X$.

<u>Automorphisms</u> are self-isomorphisms.

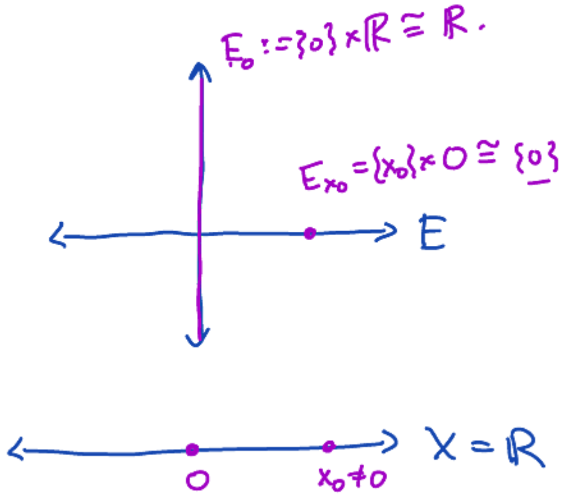
E.g., $\mathrm{Aut}(\underline{\mathbb{R}}^k) = \mathrm{Maps}(X, GL(k, \mathbb{R}))$.

vec. bdle over X

Non-example of a vector bundle (also not a fiber bundle):

(x,y) $E = \{xy=0\} \subseteq \mathbb{R}^2$ = x-axis $\cup$ y-axis

$$\begin{array}{ccc} (x,y) & & E \\ \downarrow & & \downarrow \pi_x \\ x & & \mathbb{R} \end{array}$$



(this example can be viewed as, in a suitable sense, a sheaf).

Principal bundles

G a topological group, X a space.

Def: A principal G-bundle (or a principal bundle w/ structure group G) over X is a fiber bundle $\pi : P \to X$, along with a right action of G $P \times G \to P$, such that $\pi : P \to X$ is the quotient by this action, and (implies G preserves each P_x & hence $P|_U$.)

(local triviality) $\exists$ an open cover $\mathcal{U}$ of X s.t. for every $U \in \mathcal{U}$, $\exists$ a trivialization ("local trivialization along U")

(note: G acts freely on P, & each $P_x \cong G$, as spaces w/ G action (but no canon. group str on P_x))

$$\begin{tikzcd} P|_U \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

$P|_U \circlearrowleft G$, $U \times G \circlearrowleft G$; π_U ← proj. to U; π ← map from above

which is G-equivariant; i.e., if $\varphi(p) = (z, g_0)$ then $\varphi(pg) = (z, g_0 g)$.

Obs: If $\pi : E \to X$ is a vector bundle of rank k, $\exists$ an associated principal $GL(k, \mathbb{R})$ bundle $\tilde{\pi} : P \to X$, defined as $P = \{(x, v_1, \dots, v_k) \mid x \in X, (v_1, \dots, v_k) \text{ basis for } E_x\}$.

"frame bundle", $\mathrm{Frame}(E)$.

$GL(k, \mathbb{R})$ acts on P by "change of basis" action, local triviality follows from local triviality of $E \to X$.

It turns out one can naturally go back from Frame(E) to E, as a special case of a more general construction that associates

$$\left(P := \text{principal } G \text{ bundle},\ \underset{\text{representation}}{G \to GL(V)}\right) \longmapsto P \times_G V \text{ associated vector bundle.}$$

Applying this to

$$\left(\text{Frame}(E),\ GL(k) \xrightarrow{id} GL(k)\right) \quad \text{produces } E.$$