$G \to \mathrm{Homeo}(F)$ map of top. groups

Operations on principal bundles:

$P \xrightarrow{\pi} X$ principal G-bundle, F any top. space w/ a left G action $G \times F \to F$

$\rightsquigarrow$ can form the associated fiber bundle

$$P \times_G F := P \times F /\sim \quad \text{where } (zg, f) \sim (z, gf). \quad \forall g, z, f.$$

$\pi : P \times_G F \to X$ defined by $\pi([z,f]) := \pi(z)$ (check well-defined);

fibers non-canonically isomorphic to F, & locally trivial (check: uses local triviality of P).

If the action has 'more structure', the associated fiber bundle will have more structure too.

e.g.,
- If $F = V$ a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and $G \times V \to V$ is a linear action (meaning $G \to GL(V) \subset \mathrm{Homeo}(V)$),

then $P \times_G V$ is a vector bundle of rank $= \dim(V)$, w/ fibers all (non-canonically) isomorphic to V.

- If have a map of top. groups $G \to H$ (e.g., continuous group hom.), induces an action $G \times H \to H$.

then $P \times_G H$ is a principal H-bundle.

Let's give some examples of this construction.

Note: Have the tautological action $GL(\mathbb{R}^k) \times \mathbb{R}^k \to \mathbb{R}^k$, $(T, v) \longmapsto T(v)$ $(GL(\mathbb{R}^k) \xrightarrow{id} GL(\mathbb{R}^k))$.

Claim: If $\begin{matrix} E \\ \pi\downarrow \\ X \end{matrix}$ any vector bundle $\rightsquigarrow$ $\begin{matrix} \mathrm{Frame}(E) \\ \downarrow \\ X \end{matrix}$ principal $GL(\mathbb{R}^k)$ bundle $\rightsquigarrow$ $\mathrm{Frame}(E) \times_{GL(\mathbb{R}^k)} \mathbb{R}^k$ (using this action)

Then $\mathrm{Frame}(E) \times_{GL(\mathbb{R}^k)} \mathbb{R}^k \cong E$.

In fact, (exercise): The following are inverse operations

$$(*) \quad \left\{ \begin{matrix}\text{Vector bundles of} \\ \text{rank } k \\ \text{on } X\end{matrix} \right\} \underset{(-)\times_{GL(\mathbb{R}^k)} \mathbb{R}^k}{\overset{\mathrm{Frame}}{\rightleftarrows}} \left\{ \begin{matrix}\text{Principal } GL(k,\mathbb{R}) \\ \text{bundles on } X\end{matrix} \right\}$$

tautological action.

In particular by applying (*), given a representation $GL(\mathbb{R}^k) \to GL(\mathbb{R}^m)$, we get an associated operation

$$\begin{tikzcd} \{\text{rank } k \text{ vector bundles}\} \arrow[r, dashed] \arrow[d] & \{\text{rank } m \text{ vector bundles}\} \\ \text{Principal } GL(k)\text{-bundle} \arrow[r, "\text{assoc. bundle}"'] & \text{Principal } GL(m)\text{-bundle} \arrow[u] \end{tikzcd}$$

Ex: (1) $\underset{GL(\mathbb{R}^k)}{GL(k,\mathbb{R})}$ acts on $\mathbb{R}$ by $GL(k,\mathbb{R}) \to GL(1,\mathbb{R}) = \mathbb{R}\setminus 0$

$$A \longmapsto \det(A)$$

$\rightsquigarrow$ get for any rank k $E \to X$ an associated line bundle $\det(E) \to X$. (note: this coincides w/ "$\begin{matrix}\Lambda^{top}E \\ \downarrow \\ X\end{matrix}$").

(2) Consider $GL(k,\mathbb{R})$ acting on $\mathbb{R}^k$ via

$$(A,\vec{v}) \longmapsto (A^{-1})^T \vec{v}.$$

The associated vector bundle (starting from E) is called the dual vector bundle E^*.
(similar constructions work over $\mathbb{C}$)

Other operations on vector bundles: (over $\mathbb{R}$ or $\mathbb{C}$)

- Pullback: Given a vector bundle $\begin{matrix}E \\ \downarrow\pi \\ Y\end{matrix}$ and a continuous map $f: X \to Y$, get a vector bundle $\begin{matrix}f^*E \\ \downarrow f^*\pi \\ X\end{matrix}$, along with a map (lying over f)

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$ (linear on each fiber)

by definition, $f^*E := \{(x,e) \mid f(x) = \pi(e)\} \subseteq X \times E$, and $(f^*\pi)(x,e) = x$.
("$X \times_Y E$" or "$X \times_{(f,\pi)} E$")

Note: $(f^*E)_x := E_{f(x)}$. (a vector space).

Locally trivial? (exercise).

Note: • We can also pull back principal bundles via the same construction (replace E w/ P), & the G action is inherited from G action on $X \times P \supseteq X \times_Y P = f^*P$.

• special case: $X \overset{i}{\subset} Y$ inclusion of subset, then $i^*E = E|_X$ $(= \pi^{-1}(X))$.

- Cartesian product of vector bundles (or principal bundles)

if
$$\begin{array}{cc} \overset{\text{rank } m}{E} & \overset{\text{rank } n}{F} \\ \downarrow \pi_E & \downarrow \pi_F \\ X & Y \end{array} \quad \left(\text{resp.} \begin{array}{cc} P \circlearrowleft G & Q \circlearrowleft H \\ \downarrow \pi_P & \downarrow \pi_Q \\ X & Y \end{array}\right), \text{ then}$$

$$\begin{array}{c} E \times F \\ \downarrow (\pi_E, \pi_F) \\ X \times Y \end{array} \quad \left(\text{resp.} \begin{array}{c} P \times Q \\ \downarrow (\pi_P, \pi_Q) \\ X \times Y \end{array}\right)$$
is a vector bundle of rank $m+n$. (resp. principal $G \times H$ bundle).

(exercise)

- "fiberwise direct sum" of vector bundles (whitney sum):

Given
$$\begin{array}{c} E \\ \pi_E \downarrow \\ X \end{array}, \quad \begin{array}{c} F \\ \pi_F \downarrow \\ X \end{array}, \quad \text{first take} \quad \begin{array}{c} E \times F \\ \downarrow (\pi_E, \pi_F) \\ X \times X \end{array}, \text{ then}$$

define $E \oplus F := \Delta^*(E \times F)$, where $\Delta : X \to X \times X$, $x \mapsto (x,x)$ diagonal embedding.

check: $(E \oplus F)_x := E_x \oplus F_x$.

- we can similarly define "fiberwise" operations $E \otimes F$, $\mathrm{Hom}_{\mathbb{R}}(E,F)$ (or $\mathrm{Hom}_{\mathbb{C}}$ if $\mathbb{C}$-vec. bdles); easiest way to see this is as follows:

starting with
$$\begin{array}{cc} \overset{\text{rank } m}{E}, & \overset{\text{rank } n}{F} \\ \pi_E \downarrow & \downarrow \pi_F \\ X & X \end{array}, \text{ let } \begin{array}{cc} P, & Q \\ \pi_P \downarrow & \downarrow \pi_Q \\ X & X \end{array} \text{ be associated frame bundles over } X.$$

P has structure group $G := GL(m, \mathbb{R})$

Q " " $H := GL(n, \mathbb{R})$.

Form $\Delta^*(P \times Q) := P \times_X Q$ · a principal $G \times H$ bundle over X.

Observe that $G \times H = GL(m,\mathbb{R}) \times GL(n, \mathbb{R})$ acts naturally on

- $\mathbb{R}^m \oplus \mathbb{R}^n$ by $(g,h)(v \oplus w) = gv \oplus hw$
- $\mathbb{R}^m \otimes \mathbb{R}^n$ by $(g,h)(v \otimes w)$ is $gv \otimes hw$
- $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n)$ $(g,h)(T) = h \circ T \circ (g^{-1})^T$

We call the associated bundles $E \oplus F$, $E \otimes F$, $\mathrm{Hom}_{\mathbb{R}}(E,F)$ resp. $\mathrm{Hom}_{\mathbb{C}}(-,-)$.

check: agrees w/ def'n above. The fiber at each $x \in X$ is $E_x \oplus F_x$, $E_x \otimes F_x$, $\mathrm{Hom}_{\mathbb{R}}(E_x, F_x)$ respectively.

- the dual bundle can be realized as $E^* = \mathrm{Hom}_{\mathbb{R}}(E, \underline{\mathbb{R}})$.

Def: A section of a fiber bundle $\begin{matrix} Q \\ \downarrow\pi \\ X \end{matrix}$ is a map $s: X \to Q$ with $\pi \circ s = id_X$. ✯ denoted $\begin{tikzcd} Q \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$.

✯ $\Rightarrow$ $s(x) = (x, s_x)$ where $s_x \in Q_x$

(thinking of Q ~~as~~ set theoretically as $\coprod_{x \in X} Q_x$).

Thm: A principal bundle is trivial iff it has a section.

(rmk: In contrast, while it is true a line bundle is trivial $\Leftrightarrow$ $\exists$ non-zero section, not nec. true for higher rank vec. bundles)

(E vec. bundle rank k $\rightsquigarrow$ Frame(E) is trivial iff $\exists$ a section $X \to$ Frame(E) $\rightsquigarrow$ E is trivial iff $\exists$ a k-tuple of sections which form a frame at each point x (i.e., a basis for each fiber).)

Pf: $\Rightarrow$ ✓ b/c $\begin{tikzcd} X\times G \arrow[d] \\ X \arrow[u, bend right, "s(x)=(x,id)"'] \end{tikzcd}$.

$\Leftarrow$ Say $\exists$ s: $\begin{tikzcd} P \arrow[d, "\pi"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$. Then define a map of principal bundles (i.e., φ G-equiv.)

$$\begin{tikzcd} X\times G \arrow[rr, "\varphi"] \arrow[dr, "\pi_X"'] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

, by $\varphi(x,g) = s(x)\cdot g$.

φ is automatically an iso. by next lemma. ☒

Lem: Any non-trivial morphism of G-bundles $\begin{tikzcd} P_0 \arrow[rr, "f"] \arrow[dr] & & P_1 \arrow[dl] \\ & X & \end{tikzcd}$ (i.e., G-equiv.) is an isomorphism.

Pf: Special case $P_0 = \begin{matrix} X\times G \\ \downarrow \pi_X \\ X \end{matrix}$, $P_1 = \begin{matrix} X\times G \\ \downarrow \pi_X \\ X \end{matrix}$. & $f: \begin{tikzcd} P_0 \arrow[rr] \arrow[dr] & & P_1 \arrow[dl] \\ & X & \end{tikzcd}$ $\overset{\text{equivariance}}{\Rightarrow}$ $f(x,g) = (x, g\,h(x))$ for some $h: X \to G$.

But now this map has inverse $(x,g) \longmapsto (x, g(h(x))^{-1})$.

Since a general P_0, P_1 are locally trivial, this argument applies f is an iso. in a nbhd of any x, hence everywhere. ☒.

(Rmk: In contrast, exercise:

- any vector bundle has a section, the zero section $x \longmapsto (x, 0)$.
- a line bundle is trivial iff it has a nowhere vanishing section $x \longmapsto (x, s_x)$ w/ $s_x \neq 0$ $\forall$ x.
- More generally, a rank k vec. bdle $E \to X$ is trivial iff $\exists$ a basis of sections e.g., sections $s_1, \ldots, s_k$ s.t.

$((s_1)_x, \dots, (s_k)_x)$ are a basis of E_x $\forall x \in X$.

Inner products on vector bundles: (an inner product on V is an element of $(V\otimes V)^* \ni g$ s.t. the map $\langle -,- \rangle : V\times V \to V\otimes V \to \mathbb{R}$ satisfies --)

An inner product on a vector bundle $\begin{matrix} E \\ \downarrow \\ X\end{matrix}$ is a section (g) of $(E\otimes E)^*$,

s.t. the associated pairing $\langle -,- \rangle_x$ on E_x defined by $\langle v,w\rangle_x := g_x(v\otimes w)$

is an inner product (pos definite, symmetric bilinear).

Can think of as a collection of $\langle -,- \rangle_x$ on each E_x "varying continuously" (in sense g is a continuous section)

$\Rightarrow$ if s, t are (continuous) sections, then

$x \longmapsto \langle s_x, t_x\rangle_x$ is continuous.

OR $\langle -,- \rangle \in$ "$\Gamma(\text{Bilinear}(E\times E, \mathbb{R}))$".

Lem: An inner product exists (at least if X is paracompact, i.e., admits partitions of unity) (c.f., 535a or Hatcher)

Sketch: Given a cover $\{U_\alpha\}$ over which E is loc. trivial, $\exists$ an inner product $\langle -,- \rangle_\alpha$ on each $E|_{U_\alpha}$ b/c $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ (use $\langle -,- \rangle_{\text{Euclidean}}$ on $\mathbb{R}^k$).

Then if $\{\varphi_\alpha\}$ is a partition of 1 subordinate to $\{U_\alpha\}$, we claim

$\sum \varphi_\alpha \langle -,- \rangle_\alpha$ gives an inner product on E. (exercise). $\blacksquare$.

Q: If a vector bundle comes equipped with an inner product, how can I understand this in terms of principal bundles?

Def: $P \to B$ principal G-bundle, $H \subseteq G$ subgroup. Say P has a reduction of structure group to H iff P is isomorphic to $\tilde{P} \times_H G$ for some $\tilde{P} \to B$ principal H-bundle. A choice of reduction is a choice of such $\tilde{P}$.

Lemma: Given a vector bundle $E \to B$, an inner product on E $\longleftrightarrow$ a choice of reduction of structure group of $\text{Frame}(E)$ to $O(n)$ (from $GL(n,\mathbb{R})$).

Idea: Given $\langle -,- \rangle$ on E, can consider $\text{OFrame}(E) = \{(x, v_1, \dots, v_k) \mid x \in B,\ v_1, \dots, v_k \text{ an orthonormal frame of } E_x \text{ w.r.t. } \langle -,- \rangle_x\}$.

Claim: $\text{OFrame}(E) \times_{O(n)} GL(n,\mathbb{R}) \cong \text{Frame}(E)$. (exercise).

This defines a map

$$\left\{\begin{array}{c}\text{Vector bundles}\\ \text{w/ inner product}\end{array}\right\} \underset{?}{\overset{\text{OFrame}}{\rightleftarrows}} \left\{\text{principal } O(n)\text{-bundles}\right\}$$

$P \longmapsto P \times_{O(n)} (\mathbb{R}^n, \langle -,- \rangle_{\text{Eucl.}})$ ⟵ (claim: if $G \hookrightarrow \mathbb{R}^n$ by isometries, then the Euclidean inner product on $\mathbb{R}^n$ induces an inner product on each fiber of $P \times_G \mathbb{R}^n$.)

In general many extra structures on vector bundles can be described via a reduction of structure group of the associated principal bundle.

(e.g., w/o proof: • An orientation on E^k $\longleftrightarrow$ Reduction of structure group of $\text{Frame}(E)$ to $GL^+(k,\mathbb{R}) \subseteq GL(k,\mathbb{R})$

• A complex structure on E^{2k} $\longleftrightarrow$ " " " to $GL(k,\mathbb{C}) \subseteq GL(2k,\mathbb{R})$)

real vec. bundle