

Operations on principal bundles:

$P \xrightarrow{\pi} X$  principal  $G$ -bundle,  $F$  any top. space w/ a left  $G$  action  $G \times F \rightarrow F$

$\rightarrow$  can form the associated fiber bundle

$$P \times_G F := P \times F / \sim \quad \text{where } (zg, f) \sim (z, gf). \quad \forall g, z, f.$$

$$\pi: P \times_G F \rightarrow X \quad \text{defined by } \pi([z, f]) := \pi(z) \quad (\text{check well-defined});$$

fibers non-canonically isomorphic to  $F$ , & locally trivial (check: uses local triviality of  $P$ ).

If the action has 'more structure', the associated fiber bundle will have more structure too.

- e.g.,
- If  $F = V$  a vector space (over  $\mathbb{R}$  or  $\mathbb{C}$ ) and  $G \times V \rightarrow V$  is a linear action (meaning  $G \rightarrow GL(V) \subset \text{Homeo}(V)$ ), then  $P \times_G V$  is a vector bundle of rank =  $\dim(V)$ , w/ fibers all (non-canonically) isomorphic to  $V$ .
  - If have a map of top. groups  $G \rightarrow H$  (e.g., covering group hom.),  $\leftarrow$  induces an action  $G \times H \rightarrow H$ . then  $P \times_G H$  is a principal  $H$ -bundle.

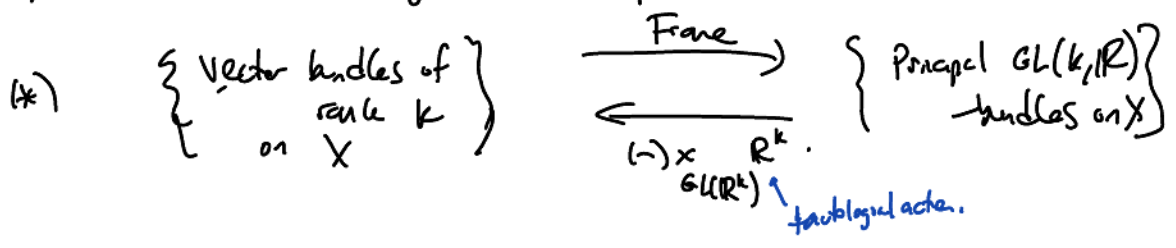
Let's give some examples of this construction.

Note: Have the fundamental action  $GL(\mathbb{R}^k) \times \mathbb{R}^k \rightarrow \mathbb{R}^k$  ( $(T, v) \mapsto T(v)$ )  $(GL(\mathbb{R}^k) \xrightarrow{id} GL(\mathbb{R}^k))$  using this action

Claim: If  $\begin{matrix} E \\ \pi \downarrow \\ X \end{matrix}$  any vector bundle  $\rightsquigarrow$   $\begin{matrix} \text{Frame}(E) \\ \downarrow \\ X \end{matrix}$  principal  $GL(\mathbb{R}^k)$  bundle  $\rightsquigarrow$   $\text{Frame}(E) \times_{GL(\mathbb{R}^k)} \mathbb{R}^k$

$$\text{Then } \text{Frame}(E) \times_{GL(\mathbb{R}^k)} \mathbb{R}^k \cong E.$$

In fact, (exercise): The following are inverse operations



In particular by applying (\*), given a representation  $GL(\mathbb{R}^k) \rightarrow GL(\mathbb{R}^m)$ , we get an associated operation  $\{\text{rank } k \text{ vector bundles}\} \dashrightarrow \{\text{rank } m \text{ vector bundles}\}$



Ex: (1)  $GL(k, \mathbb{R})$  acts on  $\mathbb{R}$  by  $GL(k, \mathbb{R}) \rightarrow GL(1, \mathbb{R}) = \mathbb{R} \setminus \{0\}$   
 $A \longmapsto \det(A)$

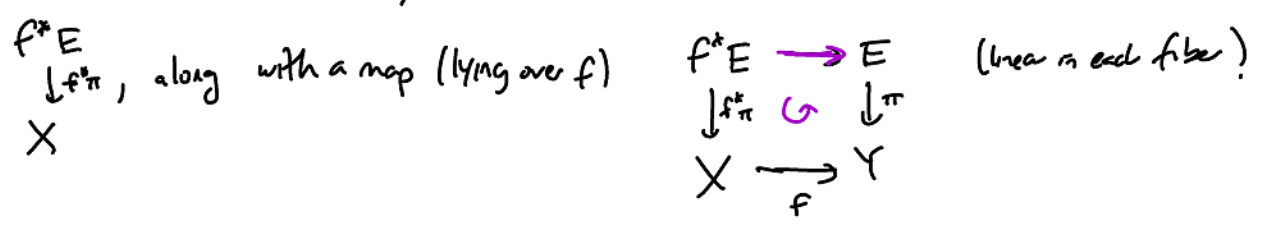
$\leadsto$  get for any rank  $k$   $E \rightarrow X$  an associated line bundle  $\det(E) \rightarrow X$ . (note: this coincides w/ " $\wedge^{\text{top}} E$ ").

(2) Consider  $GL(k, \mathbb{R})$  acting on  $\mathbb{R}^k$  via  $(A, \vec{v}) \longmapsto (A^{-1})^T \vec{v}$ .

The associated vector bundle (starting from  $E$ ) is called the dual vector bundle  $E^*$ . (similar constructions work over  $\mathbb{C}$ )

Other operations on vector bundles: (over  $\mathbb{R}$  or  $\mathbb{C}$ )

• Pullback: Given a vector bundle  $\frac{E}{\downarrow \pi}$  and a continuous map  $f: X \rightarrow Y$ , get a vector bundle



by definition,  $f^*E := \{(x, e) \mid f(x) = \pi(e)\} \subseteq X \times E$ , and  $(f^*\pi)(x, e) = x$ .  
 ("  $X \times_Y E$  " or "  $X \times_{(f, \pi)} E$  ")

Note:  $(f^*E)_x := E_{f(x)}$ . (a vector space).

Locally trivial? (exercise 1).

Note: • We can also pull back principal bundles via the same construction (replace  $E$  w/  $P$ ), & the  $G$  action is inherited from  $G$  action on  $X \times P \ni X \times_Y P = f^*P$ .

• special case:  $X \subset Y$  inclusion of subset, then  $i^*E = E|_X (= \pi^{-1}(X))$ .

• Cartesian product of vector bundles (or principal bundles)

if  $E \xrightarrow{\pi_E} X$  (rank  $m$ ) and  $F \xrightarrow{\pi_F} Y$  (rank  $n$ ) (resp.  $P \xrightarrow{\pi_P} X$  (rank  $g$ ) and  $Q \xrightarrow{\pi_Q} Y$  (rank  $h$ )), then

$E \times F \xrightarrow{(\pi_E, \pi_F)} X \times Y$  (resp.  $P \times Q \xrightarrow{(\pi_P, \pi_Q)} X \times Y$ ) is a vector bundle (resp. principal  $G \times H$  bundle) of rank  $m+n$ . (exercise)

• "fiberwise direct sum" of vector bundles (Whitney sum):

Given  $E \xrightarrow{\pi_E} X$  and  $F \xrightarrow{\pi_F} X$ , first take  $E \times F \xrightarrow{(\pi_E, \pi_F)} X \times X$ , then

define  $E \oplus F := \Delta^*(E \times F)$ , where  $\Delta: X \rightarrow X \times X$  diagonal embedding,  $x \mapsto (x, x)$

check:  $(E \oplus F)_x := E_x \oplus F_x$ .

• We can similarly define "fiberwise" operations  $E \otimes F$ ,  $\text{Hom}_{\mathbb{R}}(E, F)$ ; easiest way to see this is as follows:

starting with  $E, F$  (rank  $m, n$ ), let  $P, Q$  be associated frame bundles over  $X$ .  $P$  has structure group  $G := GL(m, \mathbb{R})$ ,  $Q$  " "  $H := GL(n, \mathbb{R})$ . (or  $\text{Hom}_{\mathbb{C}}$  if  $\mathbb{C}$ -vec. bdl's)

Form  $\Delta^*(P \times Q) := P \times_X Q$  a principal  $G \times H$  bundle over  $X$ .

Observe that  $G \times H = GL(m, \mathbb{R}) \times GL(n, \mathbb{R})$  acts naturally on

- $\mathbb{R}^m \oplus \mathbb{R}^n$  by  $(g, h)(v \oplus w) = gv \oplus hw$
- $\mathbb{R}^m \otimes \mathbb{R}^n$  by  $(g, h)(v \otimes w) = gv \otimes hw$
- $\text{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n)$  by  $(g, h)(\cdot T) = h \circ T \circ (g^{-1})^T$

We call the associated bundles  $E \oplus F$ ,  $E \otimes F$ ,  $\text{Hom}_{\mathbb{R}}(E, F)$  (resp.  $\text{Hom}_{\mathbb{C}}(-, -)$ ).  
 check: agrees w/ definition. The fiber at each  $x \in X$  is  $E_x \oplus F_x$ ,  $E_x \otimes F_x$ ,  $\text{Hom}_{\mathbb{R}}(E_x, F_x)$  respectively.  
 • the dual bundle can be realized as  $E^* = \text{Hom}_{\mathbb{R}}(E, \underline{\mathbb{R}})$ .



$((S_1)_x, \dots, (S_k)_x)$  are a basis of  $E_x \forall x \in X$ .

Inner products on vector bundles: (an inner product on  $V$  is an element of  $(V \otimes V)^*$  s.t. the map  $\langle -, - \rangle : V \times V \rightarrow V \otimes V \rightarrow \mathbb{R}$  satisfies --)

An inner product on a vector bundle  $\frac{E}{X}$  is a section  $g$  of  $(E \otimes E)^*$ ,

s.t. the associated pairing  $\langle -, - \rangle_x$  on  $E_x$  defined by  $\langle v, w \rangle_x := g_x(v \otimes w)$  is an inner product (pos definite, symmetric bilinear).

Can think of as a collection of  $\langle -, - \rangle_x$  on each  $E_x$  "varying continuously" (in sense  $g$  is a continuous section)

$\Rightarrow$  if  $s, t$  are (contn) sections, then  $x \mapsto \langle s_x, t_x \rangle_x$  is contn. OR  $\langle -, - \rangle \in \Gamma(\text{Bilinear}(E \times E, \mathbb{R}))$

LEM: An inner product exists (at least if  $X$  is paracompact, i.e., admits partitions of unity)

Sketch: Given a cover  $\{U_\alpha\}$  over which  $E$  is loc. trivial,  $\exists$  an inner product  $\langle -, - \rangle_\alpha$  on each  $E|_{U_\alpha}$  b/c  $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$  (use  $\langle -, - \rangle_{\text{Euclidean}}$  on  $\mathbb{R}^k$ ).

Then if  $\{\varphi_\alpha\}$  is a partition of 1 subordinate to  $\{U_\alpha\}$ , we claim  $\sum \varphi_\alpha \langle -, - \rangle_\alpha$  gives an inner product on  $E$ . (exercise).  $\square$

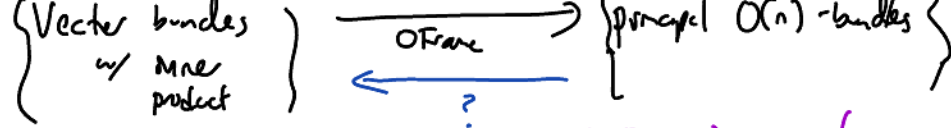
Q: If a vector bundle comes equipped with an inner product, how can I understand this in terms of principal bundles?

Def:  $P \rightarrow B$  principal  $G$ -bundle,  $H \subseteq G$  subgroup. Say  $P$  has a reduction of structure group to  $H$  iff  $P$  is isomorphic to  $\tilde{P} \times_H G$  for some  $\tilde{P} \rightarrow B$  principal  $H$ -bundle. A choice of reduction is a choice of such  $\tilde{P}$ .

Lemma: Given a vector bundle  $E \rightarrow B$ , an inner product on  $E \iff$  a choice of structure group of  $\text{Frame}(E)$  to  $O(n)$  (from  $GL(n, \mathbb{R})$ ).

Idea: Given  $\langle -, - \rangle$  on  $E$ , can consider  $O\text{Frame}(E) = \{(x, v_1, \dots, v_n) \mid x \in B, v_i \rightarrow v_n \text{ an orthonormal frame of } E_x \text{ w.r.t. } \langle -, - \rangle_x\}$

Claim:  $O\text{Frame}(E) \times_{O(n)} GL(n, \mathbb{R}) \cong \text{Frame}(E)$ , (exercise). This defines a map



$P \mapsto P \times_{O(n)} \mathbb{R}^n \xleftrightarrow{\text{Eucl.}} \mathbb{R}^n$  ← (claim: if  $G \curvearrowright \mathbb{R}^n$  by isometries, then the Euclidean inner product on  $\mathbb{R}^n$  induces an inner product on each fiber of  $P \times_G \mathbb{R}^n$ .)

In general many extra structures on vector bundles can be described via a reduction of structure group of the associated principal bundle.

- (e.g., w/o proof): • An orientation on  $E^k \iff$  Reduction of structure group of  $\text{Frame}(E)$  to  $GL^+(k, \mathbb{R}) \subseteq GL(k, \mathbb{R})$
  - A complex structure on  $E^{2k} \iff$  " " to  $GL(k, \mathbb{C}) \subseteq GL(2k, \mathbb{R})$
- $\uparrow$   
 real vec. bundle