Homotopy invariance of pullbacks

Lemma (*): If $f_0, f_1 : X \longrightarrow Y$ are homotopic maps, $E \xrightarrow{\pi} Y$ vector bundle or principal G-bundle, and say X is paracompact. Then $f_0^* E \cong f_1^* E$ as (vector/principal) bundles over X.

Rmk: Same is true for arbitrary fiber bundles.

To prove Lemma, we'll make use of an important property satisfied by such bundles over paracompact spaces, homotopy lifting property (HLP). (already appears in e.g., covering space theory)

$A \subseteq X$ subspace. "with respect to (X, A)."

Def: A map $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ satisfies HLP with respect to X (rel A) if,

given a homotopy $F : X \times I \longrightarrow B$ and a lift

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ X \times 0 \arrow[ur, "\tilde{f}_0"] \arrow[r, "f_0"'] & B \end{tikzcd}$$

$F|_{X \times 0}$

of $f_0 = F(-,0)$, (i.e., $\pi \circ \tilde{f}_0 = f_0$),

(and a lift $\tilde{G} : A \times I \longrightarrow E$ of $F|_{A \times I}$, so $\pi \circ \tilde{G} = F|_{A \times I}$, agreeing w/ $\tilde{f}_0|_A$ along $A \times 0$).

Then, $\exists$ a homotopy $\tilde{F} : X \times I \longrightarrow E$ lifting F (i.e., $\pi \circ \tilde{F} = F$) and agreeing with $\tilde{f}_0 = F(-,0)$ on $X \times 0$.

(and further agreeing with $\tilde{G}$ when restricted to $A \times I$).

What we need is:

Thm: (Hatcher prop 4.48 + references that follow):

A fiber bundle $E \xrightarrow{\pi} B$ has HLP for all (X, A) if B is paracompact.

(Hatcher proves explicitly that even if B not paracompact $E \xrightarrow{\pi} B$ has HLP for all CW pairs).

Rmk: A weaker condition than requiring $E \xrightarrow{\pi} B$ to a fiber bundle is requiring it to satisfy HLP for all CW pairs (X, A), equivalently (by iteration) for all $(D^n, \partial D^n)$ $\forall n$. This is called having a Serre fibration, & suffices for many purposes.

Proof of homotopy invariance Lemma (*) (Recall have $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$, $f_0, f_1 : X \to B$):

Let $F : X \times I \longrightarrow Y$ be the homotopy (so $f_0 = F(-,0)$, $f_1 = F(-,1)$) and consider the pullback $\begin{array}{c} F^*E \\ \downarrow \\ X\times I \end{array}$. We want to show that $\underset{\substack{\| \\ f_0^*E}}{F^*E|_{X\times\{0\}}} \cong F^*E|_{X\times\{1\}} = f_1^*E$.

Let $p: X\times I \to X$ projection to X.

It is sufficient to show
$$\begin{tikzcd} p^*f_0^*E \arrow[rr, "\cong"] \arrow[dr] & & F^*E \arrow[dl] \\ & X\times I & \end{tikzcd}$$
☆ as (vector, principal) bundles over $X\times I$.

(why? restricting to $X\times\{1\}$, we'd get: $f_0^*E \cong f_1^*E$ as desired).

[specifying the above ☆ amounts to exhibiting an iso for each $x\in X$, $t\in[0,1]$,
$$(p^*f_0^*E)_{(x,t)} \cong (F^*E)_{(x,t)} = E_{F(x,t)} = E_{f_t(x)} \quad (f_t := F(-,t)).$$
$$\begin{array}{c} (p^*f_0^*E)_{(x,t)} \\ \wr\| \\ (f_0^*E)_x \\ \| \\ E_{f_0(x)} \end{array}$$
coincides w/ $E_{f_0(x)}$ when $t=0$.

(continuously varying in x,t).]

continuously varying in x,t).

Consider the fiber bundle

- $P = \mathrm{Hom}_G(p^*f_0^*E, F^*E)$ of fiberwise maps
$\downarrow$
$X\times I$ check: principal G-bundle.
(in case E is a principal bundle;)
note a section gives a map of principal bundles $p^*f_0^*E \to F^*E$, which is automatically an iso!

OR

- $P = \mathrm{Iso}_{\mathbb{R}}(p^*f_0^*E, F^*E)$ (subbundle of $\mathrm{Hom}_{\mathbb{R}}(-,-)$ consisting of fiberwise linear isomorphisms)
$\downarrow$
$X\times I$ check: this is a principle $GL(k,\mathbb{R})$-bundle, k-rank (G).

check: this is indeed a fiber bundle, and a section gives precisely the bundle isomorphism
$$p^*f_0^*E \cong F^*E \quad \text{we want.}$$

Observe $P|_{X\times\{0\}}$ has a preferred section:
$$\begin{array}{c} P|_{X\times\{0\}} \\ \downarrow \uparrow \\ X\times 0 \end{array} \quad s: (x,0) \longmapsto (x,0,\mathrm{id})$$

In other words, the homotopy

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ X\times I \arrow[r, "id"] & X\times I \end{tikzcd}$$

has a lift id_0 along $X\times 0$.

By HLP for $P \to X\times I$ (since $X / X\times I$ are paracompact), we can therefore find a lift of id extending the lift id_0 along $X\times 0$.

$\Rightarrow p^* f_0^* E \cong F^* E \underset{\substack{\text{restrict} \\ \text{to } X\times 1}}{\Rightarrow} f_0^* E \cong f_1^* E$. ☒.

Some consequences of the homotopy invariance property:

Lemma $\Leftrightarrow$ For any $X \to Y$, the map $f^* : \{\text{principal/vector bundles on } Y\}/_{\text{iso.}} \to \{\text{principal/vec. bdls on } X\}/_{\text{iso.}}$ only depends on $[f] \in [X,Y]$.

If we denote by $Bun_G(X) := \{\text{principal } G\text{-bundles on } X\}/_{\text{iso.}}$

$Vect_k(X) := \{\text{rank } k \text{ vec. bundles on } X\}/_{\text{iso.}}$,

$\Rightarrow Bun_G(-)$ and $Vect_k(-)$ are (contravariant) "<u>homotopy functors</u>". (akin to $H^k(-)$).

In <u>particular</u>:

<u>Cor</u>: that if $f: X \to Y$ is a homotopy equivalence, then

$$[f]^* : \; Bun_G(Y) \xrightarrow{\cong} Bun_G(X)$$
$$Vect_k(Y) \xrightarrow{\cong} Vect_k(X)$$

<u>Pf</u>: Let $g: Y \to X$ htpy inverse. Then $g^* = [g]^*$ is inverse to $f^* = [f]^*$. ☒

<u>Cor</u>: Over a contractible space, any vec. bundle resp. principal bundle is trivial.

<u>Pf</u>: X contractible, and $x_0 \overset{i}{\hookrightarrow} X$ any point. Then $j: X \to x_0$ (projection) is homotopy inverse, i.e., $ij \overset{\exists \text{ homotopy}}{\simeq} id_X$ & $ji \simeq id_{x_0}$ (of course $ji = id_{x_0}$).

$\Rightarrow j^* : \underset{\{x_0\times G\}}{\underset{\shortparallel}{Bun_G(x_0)}} \xrightarrow{\cong} \underset{\{X\times G\}}{Bun_G(X)}$ ☒.

$\{x_0 \times G\} \xmapsto{\text{calculate}} \{X\times G\}$

$$\left(\text{OR} \quad \begin{tikzcd} \mathrm{Vect}_k(x_0) \arrow[r, "\cong"] \arrow[d, phantom, "\shortparallel"] & \mathrm{Vect}_k(X) \\ \{x_0 \times \mathbb{R}^k\} \arrow[r, leftrightarrow, "\text{calculate}"'] & \{X \times \mathbb{R}^k\} \end{tikzcd} \right)$$

<u>Clutching functions</u>:

$E \xrightarrow{\pi} B$ fiber bundle.

Fix a trivializing cover $\{U_\alpha\}_{\alpha\in I}$ of B, along with trivializations

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\cong"] \arrow[dr] & & E|_{U_\alpha} \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

$\psi_\alpha : U_\alpha \times F \xrightarrow{\cong} E|_{U_\alpha}$. On $U_\alpha \cap U_\beta$, comparing trivializations gives us a map over $U_\alpha \cap U_\beta$ of fiber bundles,

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times F \arrow[r, "\psi_\beta"] \arrow[rr, bend left, "(p,f) \mapsto (p,\, \phi_{\alpha\beta}(p)(f))"] \arrow[dr] & E|_{U_\alpha \cap U_\beta} \arrow[r, "\psi_\alpha^{-1}"] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[dl] \\ & U_\alpha \cap U_\beta & \end{tikzcd}$$

determined by a map $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F)$, called the <u>clutching functions</u> of E w.r.t. $\{U_\alpha\}$.

- If E is a vector bundle, by using local trivializations of E as a vector bundles, the clutching functions land in $GL(\mathbb{R}^k) \subseteq \mathrm{Homeo}(\mathbb{R}^k)$.
- principal^G bundle, clutching functions can be made to take values in $G \subset \mathrm{Homeo}(G)$ by using a cover trivializing the bundle as principal bundle.

The group the clutching fcns. take value in, $G \subset \mathrm{Homeo}(F)$, is called the structure group of the bundle.

The cover $\{U_\alpha\}$ & clutching functions in fact determine the bundle completely:

Given B, a cover $\{U_\alpha\}_{\alpha\in I}$ of B, a space F, a group G which acts on F (i.e., $G \to \mathrm{Homeo}(F)$), $\{\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ maps (satisfying conditions, e.g., $\phi_{\alpha\beta}(x)\,\phi_{\beta\alpha}(x) = \underline{\mathrm{id}}$, $\phi_{\gamma\beta}(x)\,\phi_{\beta\alpha}(x) = \phi_{\gamma\alpha}(x)$).

$\rightsquigarrow$ can form a fiber bundle

$$E = \bigcup_\alpha U_\alpha \times F \big/ \sim$$ where we identify, for each α, β, $x \in U_\alpha \cap U_\beta$, $(x, f) \in U_\alpha \times F$ with $(x, \phi_{\beta\alpha}(x)(f)) \in U_\beta \times F$.

$\text{"} \phi_{\alpha\beta}(x)^{-1}(f)$

with $\pi : E \xrightarrow[\text{projection to left factor}]{} \bigcup U_\alpha \big/ \sim$ (where identify for any $x \in U_\alpha \cap U_\beta$, $x \in U_\alpha$ with $x \in U_\beta$)

$\cong B$.

Note: (the notion of equivalence should allow one to "refine" a cover)

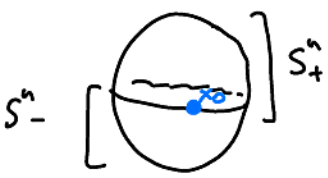
exercise: This is a fiber bundle.

In fact, up to a suitable notion of "equivalence" of such data $\{U_\alpha\}$, $\{\phi_{\alpha\beta}\}$, we can completely describe fiber bundles on B up to iso. this way.

(depending on G, F, can arrange E to be a vector resp. principal bdle)

$F = \mathbb{R}^k$, $G = GL(\mathbb{R}^k)$ — $F = G$, $G = G$.

Key example: $S^n =$



$S^n_+ \cap S^n_- \underset{\text{h.e.}}{\simeq} S^{n-1}$ (thought of as equator)

$\ni$ pick x_0 basepoint

Since each $S^n_{+/-}$ is contractible, any bundle on $S^n_{+/-}$ is trivial, moreover can choose a trivialization on either S^n_+ / S^n_- extending one on x_0 b/c $x_0 \xrightarrow{\sim} S^n_{+/-}$.

Given e.g., a complex vector bundle of rank k E on S^n,

a trivialization $E|_{x_0} \overset{\phi_{x_0}}{\cong} \mathbb{C}^k$ therefore determines trivializations

$$E_+ := E|_{S^n_+} \cong S^n_+ \times \mathbb{C}^k, \quad \text{and} \quad E_- := E|_{S^n_-} \cong S^n_- \times \mathbb{C}^k.$$

which restrict along x_0 to ϕ_{x_0}.

Using the clutching construction, can think of $E \cong E_+ \cup E_- \big/ \sim$

where along $S^n_+ \cap S^n_-$, we identify $(x, e_+) \in E_+$ with $(x, \rho(x) e_+) \in E_-$ for ~~some~~ a clutching fcn. $\rho : S^n_+ \cap S^n_- \longrightarrow GL(k, \mathbb{C})$.

Note: By construction, $\rho(x_0) = \mathrm{id} \in GL(k,\mathbb{C})$

Conversely, such a ρ determines an E, by gluing as before.

Claim: E only depends on the homotopy class of ρ

$$\in \left[(S^n_+ \cap S^n_-, x_0), (GL(k,\mathbb{C}), \mathrm{id})\right]$$
$$\parallel$$
$$\left[(S^{n-1}, x_0), (GL(k,\mathbb{C}), \mathrm{id})\right]$$
$$\cong$$
$$\pi_{n-1}(GL(k,\mathbb{C}));$$

and moreover the association $\Psi : [\rho] \longmapsto E$ induces an iso.

$$\pi_{n-1}(GL(k,\mathbb{C})) \cong \mathrm{Vect}_k^{\mathbb{C}}(S^n).$$

$E \longmapsto \rho$ as above.

Sketch: construct a map $\Phi : \mathrm{Vect}_k^{\mathbb{C}}(S^n) \to \pi_{n-1}(GL(k,\mathbb{C}))$ & check its inverse to the map $\Psi : \pi_{n-1}(GL(k,\mathbb{C})) \to \mathrm{Vect}_k^{\mathbb{C}}(S^n)$

which takes $[f]$ to $E_+ \cup E_- / (x, e_+) \sim (x, f(x) e_+)$ this only depends on $[f]$ up to iso? (exercise: sketch, if $f_0 \simeq f_1$, then using the homotopy, get a vector bundle over $S^n \times I$ restricting to the v.b. for f_0 & f_1 at $S^n \times \{0\}$ & $S^n \times \{1\}$ $\Rightarrow$ they're isomorphic).

or: equivalently show directly Ψ surjective & injective. Surjective? Above shows any $E \in \mathrm{im}(\Psi)$.
Injective? Need to know if $E_1 \cong E_2$, then clutching fns. are htpic.

only choice made up to htpy was the trivialization $E_i|_{x_0}$; but can analyze directly that any two choices lead to isomorphic outcomes.