Classifying spaces for vector bundles (w/ remarks about classifying spaces for principal bdles).

Recall: introduced $G_k(\mathbb{R}^N)$ Grassmannian of k-planes in $\mathbb{R}^N$ (similarly $G_k(\mathbb{C}^N)$), along with

$E_{taut} \to G_k(\mathbb{R}^N)$ $\quad (E_{taut} \to G_k(\mathbb{C}^N))$ rank k tautological bundle (cplx. rank in $\mathbb{C}$ case)

Let $\mathbb{R}^\infty = \bigcup_{N \geq 0} \mathbb{R}^N$ (thinking of $\mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots$, $\vec{x} \mapsto (\vec{x}, 0)$) w/ weak limit topology (meaning $A \subset \mathbb{R}^\infty$ is closed iff $A \cap \mathbb{R}^N$ $\forall N$), and define

$G_k(\mathbb{R}^\infty) := \bigcup_{N \geq 0} G_k(\mathbb{R}^N)$ (note $\underset{\emptyset \text{ if } 1<k.}{G_k(\mathbb{R}^1)} \hookrightarrow G_k(\mathbb{R}^2) \hookrightarrow \cdots$). This again comes

a tautological bundle $E_{taut} \to G_k(\mathbb{R}^\infty)$, of rank k.

Similarly have $E_{taut} \to G_k(\mathbb{C}^\infty)$.

These are the "universal" rank k (real or complex) rank k vector bundles. More precisely, we have the following in the $\mathbb{R}$ case, & completely analogous statement in $\mathbb{C}$ case:

Theorem (*) X paracompact (e.g., a CW complex). Then:

(1) For any rank k vector bundle $E \xrightarrow{\pi} X$, $E = f^* E_{taut}$ for some map $f: X \to G_k(\mathbb{R}^\infty)$. (called the 'classifying map' for E.)

(2) If we have $f_0, f_1 : X \to G_k(\mathbb{R}^\infty)$ with $f_0^* E_{taut} \cong E \cong f_1^* E_{taut}$ then $f_0 \simeq f_1$

(i.e., the classifying map f in (1) is unique up to homotopy).

In other words, the map

$$[X, G_k(\mathbb{R}^\infty)] \xrightarrow[\text{(Thm)}]{\cong} \mathrm{Vect}_k^{\mathbb{R}}(X) \quad \text{is an isomorphism.}$$
$$[f] \longmapsto [f^* E_{taut}]$$

Rmk: By considering the $GL(k)$ bundle $\mathrm{Frame}(E_{taut}) \to Gr_k(\mathbb{R}^\infty)$ or the $O(k)$-bundle $\mathrm{OFrame}(E_{taut}, \langle -,- \rangle) \to Gr_k(\mathbb{R}^\infty)$ (e.g., Euclidean metric on $\mathbb{R}^\infty$),

the theorem also implies

$$[X, Gr_k(\mathbb{R}^\infty)] \xrightarrow{\cong} \mathrm{Bun}_{GL(k,\mathbb{R})}(X) \xleftarrow{\cong} \mathrm{Bun}_{O(k)}(X)$$

(iso. b/c $O(k) \hookrightarrow GL(k, \mathbb{R})$ is a homotopy equivalence, on the vec. bundle side, this is manifested by the fact that while a vec. bdle may admit more than one $\langle -,- \rangle$, there is a contractible space of $\langle -,- \rangle$'s; hence unique up to htpy equivalence).

Q: is there an analogous result for other $\mathrm{Bun}_G(X)$'s, G another group?

Yes:

Thm (Milnor): G any top. group. there exists a classifying space for G-bundles (unique up to weak homotopy equivalence), meaning a space BG & a G-bundle $\begin{smallmatrix} EG \\ \downarrow \\ BG \end{smallmatrix}$, such that the map

"classifying space of G."

"universal G-bundle"

$$\begin{aligned} [X, BG] &\xrightarrow{\cong} \mathrm{Bun}_G(X) \quad \text{is an iso.} \\ [f] &\longmapsto [f^* EG]. \end{aligned}$$

The pair (BG, EG) is characterized by (weak) contractibility of EG.

unitary group.

In light of above, we often simply call $G_k(\mathbb{R}^\infty) =: BO(k)$, & $G_k(\mathbb{C}^\infty) =: BU(k)$.

Even more generally, we have

Thm (Brown representability): Let $F: \mathrm{Spaces}^{op} \to \mathrm{Set}$ be any contravariant homotopy functor satisfying ... (some natural conditions, satisfied for instance by $F = \mathrm{Vect}_k^{\mathbb{R}/\mathbb{C}}(-)$, $F = \mathrm{Bun}_G(-)$, $F = H^n(-;A)$).

Then, F admits a classifying space, meaning $\exists$ a space C_F (e.g., $BO(k)$ for $F = \mathrm{Vect}_k^{\mathbb{R}}(-)$), and an element $\alpha_F \in F(C_F)$ (e.g., $[E_{taut}] \in \mathrm{Vect}_k^{\mathbb{R}}(BO(k))$) such that for any X, the map

$$\begin{aligned} [X, C_F] &\xrightarrow{\cong} F(X) \\ [f] &\longmapsto [f^* \alpha_F] \end{aligned} \quad \text{is a bijection.}$$

C_F is moreover unique up to (weak) htpy equivalence.

Eilenberg-MacLane space

(Def'n: $K(A,n) :=$ the classifying space for $H^n(-;A)$; e.g.,

$\exists$ a canonical element $\alpha_{A,n} \in H^n(K(A,n); A)$ s.t.

$$\begin{aligned} [X, K(A,n)] &\xrightarrow{\cong} H^n(X; A) \\ [f] &\longmapsto [f^* \alpha_{A,n}]). \end{aligned}$$

Here we'll prove Thm (*). First,

Example applications of Thm (*).

- real line bundles: Thm says $\mathrm{Vect}_1^{\mathbb{R}}(X) \cong [X, \mathbb{RP}^\infty]$

$X = S^1$: $[S^1, \mathbb{RP}^\infty] \cong \pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$. I.e., up to iso, there are two real

• if $X = S^1$, know $[S^1, \mathbb{RP}^\infty] = \pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2$. Indeed, up to equiv. there are two real line bundles on S^1, trivial bundle, and Mobius bundle.

• if $X = S^n$, $n > 1$: $[S^n, \mathbb{RP}^\infty] = \{*\}$. (b/c maps lift to universal cover S^∞, which is contractible).

• complex line bundles are similarly classified by $[X, \mathbb{CP}^\infty]$

e.g., $[S^2, \mathbb{CP}^\infty] \cong \mathrm{Vect}_1^{\mathbb{C}}(S^2) = \mathbb{Z}$. by clutching.

& $[S^n, \mathbb{CP}^\infty] = \{*\}$ for $n \neq 2$ (also by clutching).

$\left(\overset{\text{basically}}{\Longrightarrow} \pi_k(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & k = 2 \\ \{*\} & \text{else,} \end{cases} \text{ as sets at least.} \right)$.

Pf of theorem (*):

Let $E \xrightarrow{\pi} X$ be as in theorem statement. Fix a cover $\{U_\alpha\}$ of X over which E is trivial, along w/ trivializations

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr, "\phi_\alpha", "\cong"'] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^k . \arrow[dl, "\pi_{U_\alpha}"] \\ & U_\alpha & \end{tikzcd}$$

$\boxed{\begin{array}{l} \text{Define } \eta_\alpha := \pi_{\mathbb{R}^k} \circ \phi_\alpha : E|_{U_\alpha} \longrightarrow \mathbb{R}^k. \\ \text{Note: } (\eta_\alpha)|_{E_x} : E_x \xrightarrow{\cong} \mathbb{R}^k \text{ for each } x \in U_\alpha \end{array}}$

By paracompactness, we can WLOG assume U_α countable + locally finite, & can pick a subordinate partition of unity

$$\{f_\alpha : X \to \mathbb{R}\} \text{ to } \{U_\alpha\}$$

(means: $f_\alpha : X \to [0,1]$ continuous, $\mathrm{supp}(f_\alpha) \subset U_\alpha$, and $\sum f_\alpha \equiv 1$).

well-defined b/c finite sum of non-zero #'s at each $p \in X$ (by loc. finiteness of $\{U_\alpha\}$)

Consider $f_\alpha \eta_\alpha : E \longrightarrow \mathbb{R}^k$, a map which is linear on each fiber of E. Summing these together gives:

$$(\star) \quad \Phi := \bigoplus_\alpha f_\alpha \eta_\alpha : E \longrightarrow \bigoplus_\alpha \mathbb{R}^k = \mathbb{R}^\infty$$

countable sum.

This map is continuous, linear on each fiber $E_x \subset E$, and injective on each fiber $E_x \subset E$.

(exercise)

(given $x \in X$, some $f_\beta(x) \neq 0$ and hence $f_\beta \eta_\beta : E_x \xrightarrow{\cong} \mathbb{R}^k$, so Φ is injective on E_x).

Then define

$$\begin{aligned} f : X &\longrightarrow G_k(\mathbb{R}^\infty) \\ x &\longmapsto \Phi(E_x) \end{aligned}$$

this is a k-dim'l subspace, hence gives a point in $G_k(\mathbb{R}^\infty)$, by injectivity above.

f classifies E? Observe there's a natural vector bundle map

b/c $\begin{matrix} E_{taut} \\ \downarrow \\ Gr_k(\mathbb{R}^\infty) \end{matrix}$ is a sub-bundle of $\begin{matrix} Gr_k(\mathbb{R}^\infty) \times \mathbb{R}^\infty \\ \downarrow \\ Gr_k(\mathbb{R}^\infty) \end{matrix}$.

$$\begin{tikzcd} E \arrow[rr, "\Psi"] \arrow[dr] & & f^* E_{taut} \subset X \times \mathbb{R}^\infty \arrow[dl] \\ & X & \end{tikzcd}$$

given by $\Psi(e) := (\pi(e), \Phi(e)) \subset X \times \mathbb{R}^\infty$.

as in $\star$

(check: lands in $f^* E_{taut}$).

Injective on each fiber: $\Rightarrow$ Φ induces $E \xrightarrow{\cong} f^* E_{taut}$. (note: we used Rmk that says that such a Φ is automatically a homeomorphism). This establishes (1).

(2) Say we have $f_0, f_1 : X \longrightarrow G_k(\mathbb{R}^\infty)$ with $f_0^* E_{taut} \cong E \cong f_1^* E_{taut}$.

Let $\psi_i : E \xrightarrow{\cong} f_i^* E_{taut}$ for $i = 0, 1$.

$$\begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr] & & f_i^* E_{taut} \arrow[dl] \\ & X & \end{tikzcd}$$

Again we'll think of ψ_i as coming from a (linear inj. in each fiber) map to $\mathbb{R}^\infty$ as follows:

For each $x \in X$, $(\psi_i)_x : E_x \to (f_i^* E_{taut})_x = (E_{taut})_{f_i(x)} = f_i(x) \underset{\text{subspace}}{\subset} \mathbb{R}^\infty$.

Hence ψ_i induces $\Psi_i : E \longrightarrow \mathbb{R}^\infty$ (w/ $\Psi_i |_{E_x} = (\psi_i)_x : E_x \longrightarrow \mathbb{R}^\infty$ as above) linear and injective on each fiber, for $i = 0, 1$.

(Note that Ψ_i determines f_i also by $f_i(x) := \Psi_i(E_x) \in Gr_k(\mathbb{R}^\infty)$, $i = 0, 1$).

<u>Special case:</u> Suppose for each non-zero $e \in E$, $\Phi_0(e)$ is not a negative multiple of $\Phi_1(e)$. (✩)

Then, if we set

$$\Phi_t(e) = (1-t)\Phi_0(e) + t\, \Phi_1(e) \quad \text{for } t \in [0,1], \text{ and note}$$

$\Phi_t : E \to \mathbb{R}^\infty$ continues to be injective on each fiber, so this gives

$$\begin{aligned} f_t : X &\longrightarrow G_k(\mathbb{R}^\infty), \quad \text{a homotopy } f_0 \simeq f_1 \\ x &\longmapsto \Phi_t(E_x) \end{aligned}$$

<u>General case:</u>

Observe that we have the ∞-codimension subspace maps

$$\begin{aligned} F_{odd} : \mathbb{R}^\infty &\longrightarrow \mathbb{R}^\infty \\ (x_1, x_2, x_3, \ldots) &\longmapsto (x_1, 0, x_2, 0, x_3, 0, \ldots) \end{aligned}$$

$$\begin{aligned} F_{even} : \mathbb{R}^\infty &\longrightarrow \mathbb{R}^\infty \\ (x_1, x_2, x_3, \ldots) &\longmapsto (0, x_1, 0, x_2, 0, x_3, \ldots) \end{aligned}$$

and moreover $(F_{odd})_s : (1-s)\,Id_{\mathbb{R}^\infty} + s\, F_{odd}$ remain injective for each $s \in [0,1]$.

$(F_{even})_s : (1-s)\,Id_{\mathbb{R}^\infty} + s\, F_{even}$ (including $s = 1$)

So F_{odd}, F_{even} induce

$\hat{F}_{odd}, \hat{F}_{even} : G_k(\mathbb{R}^\infty) \circlearrowleft$ with $\hat{F}_{odd} \underset{\text{by } (\widehat{F_{odd}})_s}{\simeq} id \underset{\text{by } (\widehat{F_{even}})_s}{\simeq} \hat{F}_{even}$.

Now, given general $f_0, f_1 : X \to G_k(\mathbb{R}^\infty)$ & Ψ_0 and $\Psi_1 : E \to \mathbb{R}^\infty$ as above, replace Ψ_0 by the homotopic $F_{odd} \circ \Psi_0$ and Ψ_1 by homotopic $F_{even} \circ \Psi_1$.

$\Rightarrow$ replaces f_0 by homotopic $\hat{F}_{odd} \circ f_0$ and f_1 by $\hat{F}_{even} \circ f_1$.

Now since $\underbrace{F_{odd} \circ \Psi_0(e)}_{\text{of the form } (x_1, 0, x_2, 0, \ldots)}$ (non-zero) cannot be a negative multiple of $\underbrace{F_{even} \circ \Psi_1(e)}_{\text{of the form } (0, y_1, 0, y_2, \ldots)}$, i.e., satisfies (☆), we've reduced to special case.

i.e., $f_0 \simeq \hat{F}_{odd} \circ f_0 \underset{\text{special case}}{\simeq} \hat{F}_{even} \circ f_1 \simeq f_1$, as desired. ∎