Characteristic classes

A characteristic class for real or complex vector bundles (or for real/cplx. vec. bundles of rank k) assigns to each such $E \to B$ a coh. class $c(E) \in H^*(B; R)$ some R, may depend on c. (only depends on iso. class of E B) which is natural in E in the sense that if $f: A \to B$ continuous map, we get a pullback bundle $\begin{matrix} f^*E \\ \downarrow \\ A \end{matrix}$, and $\underbrace{c(f^*E)}_{\in H^\bullet(A;R)} = f^*(\underbrace{c(E)}_{H^\bullet(B;R)})$.

$f^*: H^\bullet(B;R) \to H^\bullet(A;R)$.

By the existence of classifying maps for vector bundles, such a class c is determined on all $E \to B$ by knowing

- (if complex rank k bundles) $\hat{c} := c\left(\begin{matrix} E^{k,\mathbb{C}}_{taut} \\ \downarrow \\ BU(k) \end{matrix}\right) \in H^\bullet(BU(k); R) = H^\bullet(G_k(\mathbb{C}^\infty); R)$.

  (for any other $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$, $E = f^* E_{taut}$ for some $f: B \to BU(k)$ unique up to homotopy, so naturality forces $c(E) := f^*\hat{c}$.).

- (if real rank k bundles) $\hat{c} := c\left(\begin{matrix} E^{k,\mathbb{R}} \\ \downarrow \\ BO(k) \end{matrix}\right) \in H^\bullet(BO(k); R) = H^\bullet(G_k(\mathbb{R}^\infty); R)$.

Obs: If $E \to B$ is trivial, then $E \cong p^* \underline{\mathbb{R}}^k$ (or $p^* \underline{\mathbb{C}}^k$ if cplx. case) where $p: B \to pt$

$\Rightarrow c(E) = p^*(\underbrace{c\left(\begin{matrix} \underline{\mathbb{R}}^k \\ \downarrow \\ pt. \end{matrix}\right)}_{\in H^\bullet(pt.)})$ is trivial, in sense that it's either 0 or a non-zero multiple of unit in H^0.

$H^\bullet(pt.) = \begin{cases} R & \deg 0 \\ 0 & \text{otherwise.} \end{cases}$

We conclude if $c(E)$ is not trivial in such a sense (i.e., non-zero in some degree >0), then E cannot be a trivial bundle.

First examples:

(1) the first Stiefel-Whitney class of a real line bundle $L \to X$ (gives a class $w_1(L) \in H^1(X; \mathbb{Z}/2)$):

In $BO(1) = G_1(\mathbb{R}^\infty) = \mathbb{RP}^\infty$, there exists a unique non-zero element $h \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

Define $w_1(L_{taut} \to \mathbb{RP}^\infty) := h$. (as a ring $H^\bullet(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[h]$, $|h| = 1$).

$\Rightarrow$ for any $L \to X$ classified by $X \xrightarrow{f} \mathbb{RP}^\infty$ (i.e., $L = f^* L_{taut}$), we get a def'n

$w_1(L) := f^*(h) \in H^1(X; \mathbb{Z}/2) \underset{UCT}{=} Hom(H_1(X; \mathbb{Z}), \mathbb{Z}/2) = Hom(\pi_1(X), \mathbb{Z}/2)$

well-defined b/c f well-defined up to homotopy; no torsion in H_0; b/c $\pi_1(X)^{ab} = H_1(X)$.

Given a loop $\gamma: S^1 \to X$, $w_1(L)([\gamma]) \in \mathbb{Z}/2$ is defined as $\begin{cases} 1 & \text{if } \gamma^*L \to S^1 \text{ is non-trivial} \\ 0 & \text{if } \gamma^*L \to S^1 \text{ is trivial.} \end{cases}$

(2) The first Chern class of a complex line bundle $L \to X$ (gives a class $c_1(L) \in H^2(X;\mathbb{Z})$):

In $BU(1) = G_1(\mathbb{C}^\infty) = \mathbb{CP}^\infty$, note $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[h]$ with $|h|=2$ and in particular $H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$.

We want to declare $c_1(L^{\mathbb{C}}_{taut})$ = a generator of $H^2(\mathbb{CP}^\infty)$, but which one? (two choices, so far h is only defined as a choice of generator of H^2). The choice is a convention, but we need to fix one.

We'll use the following facts to fix an iso. $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$.

- a complex vector space $V/_{\mathbb{C}}$ of finite dimension has a canonical orientation when thought of as a real vector space:
  Namely if $v_1, \dots, v_n$ is a basis over $\mathbb{C}$ declare "complex-orientation" of $V/_{\mathbb{R}}$ to be orientation induced by $(v_1, iv_1, v_2, iv_2, \dots, v_n, iv_n)$.
  obs: If swap v_s & v_t, in real basis above need to swap (v_s, iv_s) w/ (v_t, iv_t) ⇝ even # of swaps ⇝ same orientation.
  - More generally, since $GL(n,\mathbb{C})$ is connected, the image $GL(n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{R})$ lands in a connected component of $GL(2n,\mathbb{R})$, i.e., $GL(2n,\mathbb{R})^+$. (b/c it contains Id).

- In particular, complex manifolds M carry canonical orientations of their tangent bundle $\begin{matrix} TM \\ \downarrow \\ M \end{matrix}$ (thought of as a real bundle). — pick the complex orientation for every T_pM; canonical.

- In particular, for a cpct. complex manifold Q^{2n} (← real dim. $2n$.), using equivalence between homology orientations & orientations of $\begin{matrix} TM \\ \downarrow \\ M \end{matrix}$ (omitted, but proved in many places), we deduce $\exists$ a canonical fundamental class.
  $[Q] \in H_{2n}(Q;\mathbb{Z})$.

- So $\exists$ a canonical $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^1;\mathbb{Z})$ & $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$, a canonical generator $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^\infty;\mathbb{Z})$

- Define $h \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ to be the generator with $\langle h, [\mathbb{CP}^1]\rangle = +1$.

Declare $c_1\left(\begin{matrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty \end{matrix}\right) := -h$ where h is the canonical generator above.

⇒ gives a def'n for any $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ classified by $f: X \to \mathbb{CP}^\infty$ (so $f^* L_{taut} \cong L$), as:

$$c_1(L) := f^*(-h) \in H^2(X;\mathbb{Z}).$$

Lemma: $L_1, L_2 \to X$ cplx. line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(X;\mathbb{Z})$
(and same lemma holds for w_1 in case of real line bundles w/ same proof; replace $\mathbb{CP}^\infty$ by $\mathbb{RP}^\infty$, etc.)

Pf: Say $f_i : X \to \mathbb{CP}^\infty$ classifies L_i (so $f_i^* L_{taut} = L_i$) $i=1,2$.

& define $F = (f_1, f_2) : X \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$.

Let $\pi_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ projection to ith factor, $i=1,2$, &

set $L_i^{taut} := \pi_i^* L_{taut} \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$

(Rmk: For any $\begin{matrix}E\\ \downarrow \\ A\end{matrix}$, $\begin{matrix}F\\ \downarrow \\ B\end{matrix}$, $\begin{matrix}E \boxtimes F := (\pi_A^* E) \otimes (\pi_B^* F)\\ \downarrow \\ A \times B\end{matrix}$)

Obs: $L_1 \otimes L_2 = F^*(\underbrace{L_1^{taut} \otimes L_2^{taut}}_{\text{"}L_{taut} \boxtimes L_{taut}\text{"}})$

(why? $F^*(L_1^{taut} \otimes L_2^{taut}) = F^*(L_1^{taut}) \otimes F^*(L_2^{taut})$

$= \left((f_1,f_2)^* \pi_1^* L_{taut}\right) \otimes \left((f_1,f_2)^* \pi_2^* L_{taut}\right)$

$= (f_1^* L_{taut}) \otimes (f_2^* L_{taut})$

$= L_1 \otimes L_2$.).

In $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, we know $H^\bullet(\mathbb{CP}^\infty \times \mathbb{CP}^\infty; \mathbb{Z}) \underset{\text{Künneth}}{\cong} \mathbb{Z}[h_1, h_2]$, $|h_1| = |h_2| = 2$

$h_1 := \pi_1^* h$, $h_2 := \pi_2^* h$, h canonical element as above.

which in degree 2 is $\mathbb{Z}\langle h_1 \rangle \oplus \mathbb{Z}\langle h_2 \rangle$.

Claim: $c_1(L_1^{taut} \otimes L_2^{taut}) = -h_1 - h_2$.

If true, then by Obs: $c_1(L_1 \otimes L_2) = c_1(F^*(L_1^{taut} \otimes L_2^{taut})) = F^*(-h_1 - h_2)$

$= (f_1, f_2)^*(\pi_1^*(-h) + \pi_2^*(-h)) = f_1^*(-h) + f_2^*(-h) = c_1(L_1) + c_1(L_2)$.

so we'd be done.

Pf of claim: know $c_1(L_1^{taut} \otimes L_2^{taut}) = a h_1 + b h_2$; need to pin down a & b.

restricting along $\mathbb{CP}^\infty \times pt \xhookrightarrow{i_1} \mathbb{CP}^\infty \times \mathbb{CP}^\infty$:

$i_1^* L_2^{taut} \cong \underline{\mathbb{C}}$ & $i_1^* L_1^{taut} = L_{taut}$, so $i_1^*(L_1^{taut} \otimes L_2^{taut}) \cong L_{taut}$,

and $i_1^* : H^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \to H^2(\mathbb{CP}^\infty)$.

$h_1 \longmapsto h$

$h_2 \longmapsto 0$.

so $i_1^* c_1(L_1^{taut} \otimes L_2^{taut}) = i_1^*(a h_1 + b h_2) = a h$

$\shortparallel$ $\qquad$ $\updownarrow$ $a = -1$.

$c_1(i_1^*(L_1^{taut} \otimes L_2^{taut})) = c_1(L_{taut}) = -h$.

Similarly, restricting along $pt \times \mathbb{CP}^\infty \xrightarrow{i_2} \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ & comparing $\Rightarrow b = -1$ as desired. $\boxtimes$.

Rmk: {complex line bundles X, $\otimes$} form a group (identity element: $\underline{\mathbb{C}}$, and inverse of L
(similarly for real line bundles) is $L^* := \mathrm{Hom}_{\mathbb{C}}(L, \underline{\mathbb{C}})$. exercise: verify that $L^* \otimes_{\mathbb{C}} L \cong \underline{\mathbb{C}}$).

So: c_1: {cplx line bundles, $\otimes$} $\longrightarrow H^2(X;\mathbb{Z})$ is a group homomorphism.

In fact: c_1 induces an isomorphism $(\mathrm{Vect}^1_{\mathbb{C}}(X), \otimes) \xrightarrow{\cong} H^2(X;\mathbb{Z})$, complete invariant!

We won't prove this right now, one way to see it is to understand that $\mathbb{CP}^\infty = BU(1)$ is the Eilenberg-MacLane space $K(\mathbb{Z},2)$; maps $[X, \mathbb{CP}^\infty = BU(1) = K(\mathbb{Z},2)] \xrightarrow{\cong} H^2(X;\mathbb{Z})$

$$[f] \longmapsto f^*(h).$$

(More generally, $\exists$ $K(A,n)$, & classes $\alpha \in H^n(K(A;n);A)$,
s.t. $[X, K(A,n)] \xrightarrow{\cong} H^n(X;A)$
$[f] \longmapsto f^*\alpha$.) (nice paper topic!)

(as mentioned in the last set of lecture notes).