for cplx. vec. bundles — for real vec. bundles

Higher Chern and Stiefel-Whitney classes in general

There is a completely axiomatic characterization of Chern + Stiefel-Whitney classes which we now describe:

Thm: (Stiefel-Whitney classes): $\exists$ unique characteristic classes w_i of real-vector bundles, $i \geq 1$, w/ $w_i(E) \in H^i(B; \mathbb{Z}/2)$ for $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ (vec. bdle of any rank) depending only on the iso. type of E (so $w_i : Vect_k^{\mathbb{R}}(B) \to H^i(B;\mathbb{Z}/2)$)

satisfying:

(a) (naturality): w_i are char. classes, i.e., $w_i(f^*E) = f^* w_i(E)$ any $f: A \to B$.

(b) (Whitney sum formula)

Denoting by $w(E) = 1 + w_1(E) + w_2(E) + \text{——} \in H^*(B; \mathbb{Z}/2)$ the "total Stiefel-Whitney class"; ($1 = w_0(E)$ by convention.)

(so part in degree i is $w_i(E)$)

then $\boxed{w(E_1 \oplus E_2) = w(E_1) \cup w(E_2).}$

(explicitly taking degree s parts of both sides:

$$w_s(E_1 \oplus E_2) = \sum_{\substack{i+j=s \\ i \geq 0 \\ j \geq 0}} w_i(E_1) \cup w_j(E_2)).$$

i.e., $w_2(E_1 \oplus E_2) = w_2(E_1) + w_1(E_1) \cup w_1(E_2) + w_2(E_2)$, etc.)

(c) (dimension) $w_i(E) = 0$ for $i > \text{rank}_{\mathbb{R}}(E)$.

(d) (normalization) $w_1(L_{taut} \to \mathbb{R}P^\infty)$ is the unique generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

(in fact declaring $w_1(L_{taut} \to \mathbb{R}P^1) \neq 0$ is sufficient — exercise to see this determines (d).).

Thm: (Chern classes): $\exists$ unique characteristic classes c_i of cplx-vector bundles, $i \geq 1$, w/ $c_i(E) \in H^{2i}(B; \mathbb{Z})$ for $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ (vec. bdle of any rank) depending only on the iso. type of E (so $w_i : Vect_k^{\mathbb{C}}(B) \to H^{2i}(B;\mathbb{Z})$ any k)

satisfying:

(a) (naturality): c_i are char. classes, i.e., $c_i(f^*E) = f^* c_i(E)$ any $f: A \to B$.

(b) (Whitney sum formula)

Denoting by $c(E) = 1 + c_1(E) + c_2(E) + \text{——} \in H^*(B; \mathbb{Z})$ the "total Chern class"; ($1 = c_0(E)$ by convention)

(so part in degree $2i$ is $c_i(E)$)

then $\boxed{c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).}$

(as above can extract out explicit formulae for each $c_s(E_1 \oplus E_2)$)

(c) (dimension) $c_i(E) = 0$ for $i > \text{rank}_{\mathbb{C}}(E)$.

(d) (normalization) $c_1(L_{taut} \to \mathbb{C}P^\infty)$ is the generator $-h \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ where h is the canonical element specified above.

(as before, it would have sufficed to fix $c_1(L_{taut} \to \mathbb{C}P^1)$).

Observations:

- by naturality $c_i(\underline{\mathbb{C}}^k) = 0$ for any $k, i>0$ (resp. $w_i(\underline{\mathbb{R}}^k) = 0$, $i>0$)

  so $c(E \oplus \underline{\mathbb{C}}^k) = c(E) \cup c(\underline{\mathbb{C}}^k) = c(E) \cup (\underbrace{\mathbb{1}}_{\text{"}c_0(\underline{\mathbb{C}}^k)\text{"}}) = c(E)$.

  i.e., $c_j(E \oplus \underline{\mathbb{C}}^k) = c_j(E)$.

  & similarly $w_j(E \oplus \underline{\mathbb{R}}^k) = w_j(E)$.

- If a cplx rank k vec. bdle $E \cong L_1 \oplus \cdots \oplus L_k$ line bundles, then

$$\underbrace{c(E) = c(L_1) \cup \cdots \cup c(L_k)}_{\text{(Whitney sum axiom)}} \quad \text{but } c(L_k) = \mathbb{1} + c_1(L_k) \text{ by dimension axiom}$$

$\Rightarrow c_j(E)$ is determined by $c_1(L_i)$ $\forall i$.

This forces a def'n of $c_j(E)$ for such bundles (if one wants axioms to be satisfied).

If every $E \cong L_1 \oplus \cdots \oplus L_k$ then we could use this to define $c_j(E)$ $\forall j, E$.

However, not every E splits as a direct sum of line bundles.

(same discussion holds for w (real vec. bdles which split into line bundles))

We'll approach the construction of Chern + Stiefel Whitney classes by appealing to the Leray-Hirsch theorem, a tool for understanding cohomology of fiber bundles $\begin{matrix} F \to P \\ \downarrow \\ B \end{matrix}$ under some hypotheses; applied to $\begin{matrix} \mathbb{P}(E) \\ \downarrow \\ B \end{matrix}$ ($\leftarrow$ real or cplx. fiberwise 'projectivization' of E).

Recall that if $\begin{matrix} F \to E \\ \downarrow \pi \\ B \end{matrix}$ is a fiber bundle, then $\pi^*: H^\bullet(B;R) \to H^\bullet(E;R)$ is a ring map, equips $H^\bullet(E;R)$ w/ structure of a $H^\bullet(B;R)$-module ($b \in H^\bullet(B;R)$ acts by $b \cdot \underset{\in H^\bullet(E;R)}{x} := \pi^*(b) \cup x$).

Thm: (Leray-Hirsch theorem): Say $F \xrightarrow{i} E \xrightarrow{\pi} B$ a fiber bundle, & R ring s.t.

(a) $H^k(F;R)$ free & finitely generated over R for each k.

(b) The restriction map $i^*: H^\bullet(E;R) \longrightarrow H^\bullet(F;R)$ is surjective.

Under the hypotheses of (a)+(b), we can choose a splitting $c: H^\bullet(F;R) \to H^\bullet(E;R)$ (not induced by a map of spaces), i.e., for any basis $\{\gamma_j \in H^{k_j}(F;R)\}_j$ of $H^\bullet(F)$ as an R-module

we obtain classes $c_j := c(\gamma_j) \in H^{k_j}(E;R)$ which restrict to the given basis $\{\gamma_j\}$. Call such a collection $\{c_j\}$ (or the map c) a cohomology extension of the fiber.

Then, the map

$$\Phi : H^*(B;R) \otimes_R H^*(F;R) \longrightarrow H^*(E,R)$$

depends on the choice of coh. extension of fiber.

$$\sum b_i \otimes \gamma_j \longmapsto \sum \pi^*(b_i) \cup c_j$$

"$b_i \cdot c_j$" in terms of module action of $H^\bullet(B)$ on $H^\bullet(E)$.

is an isomorphism (as $H^\bullet(B;R)$-modules).

In other words, every $c \in H^\bullet(E;R)$ can be written uniquely as $\sum \pi^*(a_j) \cup c_j$ for some unique $a_j \in H^\bullet(B;R)$.

Rmks/examples:

- For a trivial fiber bundle $E = B \times F$, w/ $H^\bullet(F;R)$ free & finitely generated, have $E \xrightarrow{\pi_F} F$, & the image of $\pi_F^*: H^\bullet(F) \to H^\bullet(E)$ gives a splitting of $i^*: H^\bullet(E) \to H^\bullet(F)$. Hypotheses therefore apply, & can use $c_j := \pi_F^*(\gamma_j)$ for a given basis $\{\gamma_j\}$ of $H^\bullet(F)$. L-H for these particular c_j's is just Künneth. (Künneth: any $c \in H^\bullet(B \times F)$ can be written as $\sum \pi^*(\underset{H^\bullet(B)}{a_j}) \cup \pi^*(\underset{H^\bullet(F)}{\gamma_j})$)

  L-H is more general in a sense, as it allows other choices of c_j (but this can also be extracted from Künneth).

- unlike Künneth, L-H theorem does not assert that $H^\bullet(E) \cong H^\bullet(B) \otimes H^\bullet(F)$ as rings! This can be false. (all one gets is that $H^\bullet(B) \otimes H^\bullet(F) \cong H^\bullet(E)$ as $H^\bullet(B)$-modules).

  (alg. example: $S = k[\underset{\deg 3}{x}]/x^5$, $T = k[\underset{\deg 2}{y}]/y^2$, now there's an iso. of S-modules

  $$k[x,y]/x^5, y^2 \cong S \otimes T \cong k[\underset{\deg 3}{x}, \underset{\deg 2}{y}]/x^5, y^2 - 1 \quad \text{but not as rings!})$$

  $$x \longmapsto x$$
  $$y \longmapsto y.$$

- Example where L-H theorem fails to apply:

  Look at the Hopf bundle

  $$\underset{F}{S^1} \to \underset{=E}{S^3}, \quad S^3 \to \underset{=B}{S^2}.$$

  Then $H^\bullet(S^3)$ cannot surject on $H^\bullet(S^1)$ as a graded R-module, b/c $H^1(S^1) \cong R$, but $H^1(S^3) = 0$.

We'll postpone a discussion of the proof of L-H for now (& may partially omit it).

for cplx. vec. bundles

Construction of Chern classes, using Leray-Hirsch theorem. (+Stiefel-Whitney classes — analogous) indicated in red

$E \xrightarrow{\pi} B$ complex rank k vector bundle (resp. real vec. bundle $E \to B$

Form $\mathbb{CP}(E)$ or $\mathbb{P}(E)$ (when "$\mathbb{C}$" implicit), (analogously $\mathbb{RP}(E) \to B$, sometimes also denoted $\mathbb{P}(E)$ if $\mathbb{R}$ is implicit).

$$\begin{array}{c}\mathbb{CP}(E)\\ \downarrow\\ B\end{array}$$

"complex fibrewise projectivization of E." This is an associated fibre bundle w/ fiber $\mathbb{P}(\mathbb{C}^k) \cong \mathbb{CP}^{k-1}$.

can construct either as $(E \setminus 0_B)/\mathbb{C}^*$ or $\mathbb{C}\text{-Frame}(E) \times_{GL(k,\mathbb{C})} \mathbb{CP}^{k-1}$.

Each fiber $\mathbb{P}(E)_b \cong \mathbb{CP}(\underbrace{E_b}_{\text{cplx. vector space}}) = (E_b \setminus 0)/\mathbb{C}^*$.

There's a tautological line bundle over $\mathbb{P}(E_b)$ for each $b \in B$ as usual: $L_b^{taut} = \{(x,v) \mid x \in \mathbb{P}(E_b), \text{i.e. } x \subseteq E_b \text{ line}, \text{ and } v \in x\}$,

which assemble to give a tautological complex line bundle over $\mathbb{P}(E)$:

$$L := \{(x,v) \mid x \in \mathbb{P}(E) = \coprod_b \mathbb{P}(E_b),\ v \in x\} \qquad L \longrightarrow \mathbb{P}(E)$$
$$= \{(b,y,v) \mid b \in B,\ y \in \mathbb{P}(E_b),\ v \in y\}. \qquad (x,v) \longmapsto x.$$

So, there's a class $h_p \in H^2(\mathbb{P}(E);\mathbb{Z})$ $h_p := -\underset{\substack{\uparrow\\ \text{1st chern class}\\ \text{as previously defined.}}}{c_1^{old}}(L) \overset{\text{explicitly}}{=} f^*h$, where $f: \mathbb{P}(E) \to \mathbb{CP}^\infty$ classifies L so $f^*L_{taut} = L$, and $h \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ canon. generator.

(in the real case, similarly have tautological real line bundle $L \to (\mathbb{R})\mathbb{P}(E)$, inducing a class $h_p = w_1^{old}(L) = f^*h \in H^1(\mathbb{P}(E);\mathbb{Z}/2)$, where $f: \mathbb{P}(E) \to \mathbb{RP}^\infty$ classifies L, $h \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2)$ non-zero element).
($\mathbb{Z}/2$ coeffs, $+1 = -1$)

Now, consider $\underbrace{1 = h_p^0, h_p, h_p^2, \ldots, h_p^{k-1}}_{\text{(i.e., pullback)}} \in H^*(\mathbb{P}(E);\mathbb{Z})$. ($\text{rank}_{\mathbb{C}}(E) = k$)

Observe the restriction of $L \to \mathbb{P}(E)$ to a fiber $\mathbb{P}(E_b)$ is $L_{taut} \to \mathbb{P}(E_b) \cong L_{taut} \to \mathbb{CP}^{k-1}$.

Therefore by naturality of c_1^{old}, h_p restricts to $-c_1^{old}(L_{taut} \to \mathbb{P}(E_b)) = h \in H^2(\mathbb{CP}^{k-1};\mathbb{Z})$.

So $1, h_p, \ldots, h_p^{k-1}$ restrict to $1, h, h^2, \ldots, h^{k-1}$ the standard generators for $H^*(\mathbb{CP}^{k-1};\mathbb{Z})$ as a $\mathbb{Z}$-module. (Recall as a ring $H^*(\mathbb{CP}^{k-1}) \cong \mathbb{Z}[h]/h^k$, so $H^{2i}(\mathbb{CP}^{k-1}) = \begin{cases} \mathbb{Z}\langle h^i \rangle & 0 \le i \le k-1 \\ 0 & \text{else.}\end{cases}$
& $H^{odd}(\mathbb{CP}^{k-1}) = 0$.

So in particular $\mathbb{P}(E) \to B$ satisfies hypotheses of Leray-Hirsch; it follows that

$1, h_p, \ldots, h_p^{k-1}$ generate $H^*(\mathbb{P}(E);\mathbb{Z})$ as a $H^*(B;\mathbb{Z})$-module.

(module action: $\underset{H^*(B)}{b} \cdot \underset{H^*(\mathbb{P}(E))}{e} := \pi^*(b) \cup e$).

So every element $e \in H^*(\mathbb{P}(E))$ can be written as $\sum_{j=0}^{k-1} \pi^*(b_j) \cup h_p^j$ for unique $b_j \in H^*(B;\mathbb{Z})$.

$H^{2k}(\mathbb{P}(E);\mathbb{Z})$.

Consider the element h_p^k. (note: if $E \to B$ trivial bundle, then $\begin{array}{c} E = \mathbb{C}^k \times B \\ \downarrow \\ B \end{array}$ so $\begin{array}{c} \mathbb{P}(E) = \mathbb{C}P^{k-1} \times B \\ \downarrow \\ B \end{array}$, & $L \to \mathbb{P}(E)$ is $\pi^*_{\mathbb{C}P^{k-1}}(L_{taut})$ where $\pi_{\mathbb{C}P^{k-1}} : \mathbb{P}(E) \to \mathbb{C}P^{k-1}$ exists when E is trivial. In that case $h_p = \pi^*_{\mathbb{C}P^{k-1}} h$, and $h_p^k = \pi^*_{\mathbb{C}P^{k-1}}(h^k) \equiv 0$).

Leray-Hirsch $\Rightarrow$ there exists a relation of the form

$$(\star) \quad h_p^k + \pi^*(a_1) \cup h_p^{k-1} + \cdots + \pi^*(a_k) \cup h_p^0 = 0.$$

for unique classes $a_1 \in H^2(B;\mathbb{Z})$, $a_2 \in H^4(B;\mathbb{Z}), \ldots, a_k \in H^{2k}(B;\mathbb{Z})$.

Def: (i^{th} Chern class) $c_i(E) := a_i$ as given above, $\in H^{2i}(B;\mathbb{Z})$.

(By convention $c_0(E) = 1$, coeff. of h_p^k in relation above; & note $c_i(E) = 0$ for $i > \mathrm{rank}_{\mathbb{C}}(E)$).

Since $h_p^k \equiv 0$ when E is trivial $\Rightarrow$ each a_i hence $c_i(E) = 0$).

(real case: Have $h_p \in H^1((\mathbb{R})\mathbb{P}(E);\mathbb{Z}/2)$. Leray-Hirsch using $1, h_p, \ldots, h_p^{k-1}$ ($k = \mathrm{rank}_{\mathbb{R}}(E)$) applies, so $\exists!$ classes $a_i \in H^i(B;\mathbb{Z}/2)$ so that $h_p^k + \pi^*(a_1) \cup h_p^{k-1} + \cdots + \pi^*(a_k) \cup h_p^0 = 0$.

$\Rightarrow$ define i^{th} Stiefel-Whitney class $w_i(E) := a_i \in H^i(B;\mathbb{Z}/2)$.)