Last time:

We gave a construction of Chern classes of a complex vector bundle (resp. Stiefel-Whitney classes of a real vec. bundle), using Leray-Hirsch theorem.

(To recap: $E \to B$ cplx vec. bundle of rank, $\rightsquigarrow$ $\mathbb{P}(E) \to B$ fiberwise cplx projectivize, & $\exists$ canonical coh. class $h_p \in H^2(\mathbb{P}(E);\mathbb{Z})$. $\left(= -c_1^{old}\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{P}(E)\end{smallmatrix}\right)\right)$. The Chern classes $c_i(E) \in H^{2i}(B;\mathbb{Z})$ are the unique classes a_i s.t.

$$h_p^k + \pi^*(a_1) \cup h_p^{k-1} + \cdots + \pi^*(a_k) \cup h_p^0 = 0.)$$ (analogously for w_i).

Need to check: Whitney sum formula, naturality, also $c_1 = c_1^{old}$; in particular $c_1\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty\end{smallmatrix}\right) = -h \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$.

(real case: Have $h_p \in H^1((\mathbb{R})\mathbb{P}(E);\mathbb{Z}/2)$. Leray-Hirsch using $1, h_p, \ldots, h_p^{k-1}$ applies, so ($k = \text{rank}_{\mathbb{R}}(E)$)

$\exists!$ classes $a_i \in H^i(B;\mathbb{Z}/2)$ so that $h_p^k + \pi^*(a_1) \cup h_p^{k-1} + \cdots + \pi^*(a_k) \cup h_p^0 = 0$.

$\Rightarrow$ define ith Stiefel-Whitney class $w_i(E) := a_i \in H^i(B;\mathbb{Z}/2)$.).

Properties: (real case parallel - exercise)

Naturality? Say $f: A \to B$ with $E \to B$ & consider $f^*E \to A$.

Note that $\mathbb{P}(f^*E) = f^*\mathbb{P}(E)$, and we have a map

$$\begin{array}{ccc} f^*\mathbb{P}(E) & \xrightarrow{f} & \mathbb{P}(E) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}, \quad \& \quad \begin{array}{c} L \\ \downarrow \\ \mathbb{P}(f^*E)\end{array} = f^*\left(\begin{array}{c} L \\ \downarrow \\ \mathbb{P}(E)\end{array}\right).$$

So h_p in $H^2(\mathbb{P}(f^*(E)))$ is $f^* h_p$ from $H^2(\mathbb{P}(E))$.

$\Rightarrow$ applying f^* to ($\ast$) gives in $H^*(\mathbb{P}(f^*E))$ the following relation:

$$\underset{f^*h_p}{\overset{\shortparallel}{h_p^k}} + \pi^*(f^*a_1) \cup h_p^{k-1} + \cdots + \pi^*(f^*a_k) \cup h_p^0 = 0.$$

we conclude $c_i(f^*E) = f^*a_i = f^*c_i(E)$. ✓.

Does this recover the usual definition when $k=1$?

$L \to B$ line bundle (complex), i.e., L_b is 1-dim'l, and $\mathbb{P}(L_b)$ is a point.

i.e., $\pi : \mathbb{P}(L) \xrightarrow{\cong} B$ is a homeomorphism w/ fibers $\mathbb{CP}^0 = \text{point}$.

And moreover the tautological bundle $L_{taut} \to \mathbb{P}(L)$ corresponds under homeo. (meaning $\cong \pi^*$ of) to $L \to B$ we started with.

$$\Rightarrow h_p := \underset{\substack{\uparrow \\ H^2(\mathbb{P}(E);\mathbb{Z})}}{-c_1^{old}(L_{taut})} = -c_1^{old}(L) \in H^2(B;\mathbb{Z})$$

In $H^*(\mathbb{P}(E);\mathbb{Z})$, $h_p^0 = 1$ is a basis for $H^*(\mathbb{P}(E);\mathbb{Z})$ as a $H^*(B;\mathbb{Z})$ module.

So have a relationship

$$h_p^{\overset{\text{rank}(L)}{1}} + \overset{\text{an iso.}}{\pi^*}(c_1^{new}(L)) \smile \overset{1}{h_p^0} = 0 \quad \text{for some class } c_1^{new}(L) \in H^2(B;\mathbb{Z}).$$

$$\Rightarrow \pi^* c_1^{new}(L) = -h_p = c_1^{old}(L_{taut}) = \pi^* c_1^{old}(L)$$

$$\Rightarrow c_1^{new}(L) = c_1^{old}(L). \quad ✓$$

Whitney sum formula? (real case parallel again)

Say have E_1, E_2 complex vector bundles over B of complex ranks k, l respectively.

Form $\begin{array}{c} E_1 \oplus E_2 \\ \downarrow \\ B \end{array}$, which has sub-bundles $E_1, E_2 \subseteq E_1 \oplus E_2$ whose fibers are complementary vector spaces,

inducing $\mathbb{P}(E_1), \mathbb{P}(E_2) \hookrightarrow \mathbb{P}(E_1 \oplus E_2)$ and $\mathbb{P}(E_1) \cap \mathbb{P}(E_2) = \emptyset$.

(if V_1, V_2 complementary vector subspaces of V then $\mathbb{P}(V_1) \cap \mathbb{P}(V_2)$ is empty in $\mathbb{P}(V)$)

Let $U_1 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_1)$ $\qquad$ $U_2 = \mathbb{P}(E_1 \oplus E_2) - \mathbb{P}(E_2)$

claim: open set retracting onto $\mathbb{P}(E_2)$ + open set retracting onto $\mathbb{P}(E_1)$.

(why? $\mathbb{CP}^k \setminus \mathbb{CP}^i$ retracts onto $\mathbb{CP}^{k-i-1}$
$[x_0 : \dots : x_k]$ $\quad$ $[x_0 : \dots x_i : 0 \dots 0]$
$[0 : \dots : 0 : x_{i+1} : \dots x_k]$)

Also, L_{taut} on $\mathbb{P}(E_1 \oplus E_2)$ restricts to L_{taut} on each $\mathbb{P}(E_i)$.

$$\Rightarrow h_{\mathbb{P}(E_1 \oplus E_2)} \Big|_{\mathbb{P}(E_i)} = h_{\mathbb{P}(E_i)}.$$

rank(E_1) $\qquad$ using $c_0(E_i) = 1$ $\qquad$ rank(E_2)

$$\text{Let } \omega_1 = \sum_{j=0}^{k} \pi^* c_j(E_1) \smile h^{k-j}_{\mathbb{P}(E_1 \oplus E_2)} \qquad \omega_2 = \sum_{j=0}^{l} \pi^* c_j(E_2) \smile h^{l-j}_{\mathbb{P}(E_1 \oplus E_2)}$$

$h^{k}_{\mathbb{P}(E_1\oplus E_2)} + \pi^* c_1(E_1) \cup h^{k-1}_{\mathbb{P}(E_1\oplus E_2)} + \cdots$ $\qquad$ $h^{\ell}_{\mathbb{P}(E_1\oplus E_2)} + \pi^* c_1(E_2) \cup h^{\ell-1}_{\mathbb{P}(E_1\oplus E_2)} + \cdots$

By definition, $\omega_1|_{\mathbb{P}(E_1)} \equiv 0$ ($h_{\mathbb{P}(E_1\oplus E_2)}$ restricts to $h_{\mathbb{P}(E_1)}$) ; similarly $\omega_2|_{\mathbb{P}(E_2)} \equiv 0$.

So, ω_1 induces a class $\tilde{\omega}_1 \in H^{2k}(\mathbb{P}(E_1\oplus E_2), \mathbb{P}(E_1)) \cong H^{2k}(\mathbb{P}(E_1\oplus E_2), U_2)$.

$\mathbb{P}(E_1) \simeq U_2$
$\cong$
$\mathbb{P}(E_1\oplus E_2)\setminus\mathbb{P}(E_2)$

Also, ω_2 induces a class $\tilde{\omega}_2 \in H^{2\ell}(\mathbb{P}(E_1\oplus E_2), \mathbb{P}(E_2))$

$$\cong$$

$$H^{2\ell}(\mathbb{P}(E_1\oplus E_2), U_1).$$

Using the relative version of the cup product, $\exists$ a com. diagram

$U_1 \cup U_2 = \mathbb{P}(E_1\oplus E_2)$
$(\mathbb{P}(E_1)\cap\mathbb{P}(E_2) = \emptyset)$.
so this group is 0!

$$\begin{tikzcd} \exists\ \tilde{\omega}_1,\ \tilde{\omega}_2 \in H^{2k}(\mathbb{P}(E_1\oplus E_2), U_2) \times H^{2\ell}(\mathbb{P}(E_1\oplus E_2), U_1) \arrow[r, "\cup"] \arrow[d] & H^{2k+2\ell}(\mathbb{P}(E_1\oplus E_2), U_1\cup U_2) \arrow[d] \\ H^{2k}(\mathbb{P}(E_1\oplus E_2)) \times H^{2\ell}(\mathbb{P}(E_1\oplus E_2)) \arrow[r, "\cup"] & H^{2k+2\ell}(\mathbb{P}(E_1\oplus E_2)) \end{tikzcd}$$

so $\tilde{\omega}_1 \cup \tilde{\omega}_2 = 0$

$\omega_1, \omega_2 \longmapsto \omega_1 \cup \omega_2 = ?$

image of $\tilde{\omega}_1 \cup \tilde{\omega}_2$.

$= 0$,

So $\omega_1 \cup \omega_2 = 0$; expanding this out we get a relation:

$h_{\mathbb{P}}^{k+\ell} + \cdots\cdots = 0$ $\quad$ coming from cupping $\omega_1 \cup \omega_2$.

coeff. of $h_{\mathbb{P}}^{k+\ell-j}$ is $\sum \pi^*(c_i(E_1) \cup c_{j-i}(E_2))$.

$\Rightarrow$ $c_j(E_1\oplus E_2) = \sum_i c_i(E_1) \cup c_{j-i}(E_2)$ as desired. $\boxtimes$.

It remains to complete the axiomatic characterization of Chern classes. (resp. same for Stiefel-Whitney).

<u>Uniqueness</u>? We have classes c_i as constructed above. Say we are given $\tilde{c}_1, \ldots, \tilde{c}_i$ other char. classes which satisfy the axioms. $\star$.

Since $\tilde{c}_1\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty \end{smallmatrix}\right) = -h = c_1\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty \end{smallmatrix}\right)$, and $\tilde{c}_i\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty \end{smallmatrix}\right) = 0 = c_i\left(\begin{smallmatrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty \end{smallmatrix}\right)$ for $i \geq 2$,

$\Rightarrow \tilde{c}_i = c_i$ for all i for $\begin{matrix} L_{taut} \\ \downarrow \\ \mathbb{CP}^\infty \end{matrix}$ $\underset{\text{(naturality)}}{\Rightarrow}$ $c_i = \tilde{c}_i$ for all i for all complex line bundles $\begin{matrix} L \\ \downarrow \\ B \end{matrix}$.

$\left\{ \begin{matrix} \text{of course } c_i = \tilde{c}_i = 0 \text{ if } i > 1 \\ \text{Content: } c_1(L) = \tilde{c}_1(L) \end{matrix} \right\}$.

$\Rightarrow c(L) = \tilde{c}(L)$ where $c = \overbrace{1}^{c_0 \text{ by convention}} + c_1 + c_2 + \cdots$, $\tilde{c} = \overbrace{1}^{\text{by convention, } \tilde{c}_0} + \tilde{c}_1 + \cdots$. 'total Chern class'

$\Rightarrow$ If a complex line bundle E can be written as a direct sum $E = L_1 \oplus \cdots \oplus L_k$ of line bundles, then Whitney sum formula implies:

$$c(E) = 1 + c_1(E) + \cdots \underset{\substack{\text{Whitney} \\ \text{sum for} \\ c}}{=} \prod_{i=1}^{k} c(L_i) \underset{\substack{\text{by} \\ \text{above} \\ c(L_i) = \tilde{c}(L_i) \text{ for line bundles}}}{=} \prod \tilde{c}(L_i) \underset{\substack{\text{Whitney} \\ \text{sum for} \\ \tilde{c}}}{=} \tilde{c}(E).$$

Problem: A given vector bundle E need not admit such a decomposition.

(e.g., over S^4, the clutching construction tells us that $\mathrm{Vect}_2^{\mathbb{C}}(S^4) \underset{\substack{\text{existence} \\ \text{of metrics}}}{\cong} \mathrm{Vect}_2^{\text{Hermitian}, \mathbb{C}}(S^4)$

$\cong [S^3, \underbrace{U(2)}_{\text{structure group for a Hermitian rank 2 bundle.}}] \underset{\substack{\text{direct} \\ \text{computation, omit.}}}{\cong} \mathbb{Z}$, i.e., $\exists$ non-trivial rank-2 cplx vec. bundles.

On the other hand, we've previously seen that $\mathrm{Vect}_1^{\mathbb{C}}(S^4) \underset{\text{(by same argument)}}{=} [S^3, S^1 = U(1)] = \{*\}$.

So a non-trivial $E \xrightarrow{\text{rank 2}} S^4$ doesn't decompose).

However, we can appeal to the following powerful principle:

Prop: (Splitting principle) (we'll state for cplx vec bundles, but real case analogous w/ 'same' proof).

Given any X (paracompact), any complex v.b. $E \to X$, $\exists$ a space Z and a map $s: Z \to X$ such that

(a) $s^* E \to Z$ is isomorphic to a direct sum of line bundles.

(b) $s^*: H^*(X; \mathbb{Z}) \to H^*(Z; \mathbb{Z})$ is injective.

(statement for real vector bundles involves injectivity of s^* on $H^*(-; \mathbb{Z}/2)$).

Using the splitting principle: Say E any rank k vector bundle $\to B$. Fix an $s: Z \to B$ as in splitting principle, so $s^* E \cong L_1 \oplus \cdots \oplus L_k$. Then, we see that if $\{c_i\}$, $\{\tilde{c}_i\}$ any two systems of "Chern classes" (satisfying axioms), then:

$$
\begin{array}{ccc}
\tilde{c}_i(s^*E) & = & c_i(s^*E) \\
\| \text{ naturality} & & \| \text{ naturality} \\
s^*\tilde{c}_i(E) & & s^*c_i(E)
\end{array}
$$

by argument above, b/c $c_i = \tilde{c}_i$ on any vector bundle which splits into line bundles & s^*E splits.

We learn $s^*\tilde{c}_i(E) = s^*c_i(E)$. Since s^* is injective, $\Rightarrow \tilde{c}_i(E) = c_i(E)$. Uniqueness ✓.

To recap: so far we've constructed (modulo Leray-Hirsch theorem) Chern / Stiefel-Whitney classes & checked they satisfy the axioms; we've also shown (modulo the splitting principle) that any two constructions of these classes satisfying the axioms are the same.