Start of today: discussion about proofs of Leray-Hirsch thm & splitting principle

Leray-Hirsch theorem: Given a fiber bundle $F \overset{i}{\to} P \overset{\pi}{\to} B$, s.t. $H^k(F;R)$ free & fin gen. $\forall k$ and with $i^*: H^\bullet(P) \twoheadrightarrow H^\bullet(F)$, a choice of splitting

$$\begin{tikzcd} H^\bullet(P) \arrow[r, two heads] & H^\bullet(F) \arrow[l, bend left, "s"] \end{tikzcd}$$

(called 'coh. extension of fiber') induces a map

$$H^\bullet(B) \otimes H^\bullet(F) \xrightarrow{\Phi} H^\bullet(P)$$

(by linearly extending $(b, f) \longmapsto \pi^* b \cup s(f)$)

which is an iso. of $H^\bullet(B)$-modules.

Rmk: One corollary of this is that $\pi^*: H^\bullet(B) \to H^\bullet(P)$ must be injective.

why?

$$\begin{tikzcd} H^\bullet(B) \arrow[d, equal] \arrow[ddr, "\pi^*"] & \\ H^\bullet(B)\otimes 1 \arrow[d, hook] & \\ H^\bullet(B) \otimes H^\bullet(F) \arrow[r, "\Phi", "\cong"'] & H^\bullet(P) \end{tikzcd}$$

$\Rightarrow \pi^* = \Phi\big|_{H^\bullet(B)\otimes 1}$, but Φ is an iso. so π^* injective.

We'll use this later (in splitting principle).

Proof of the Leray-Hirsch theorem, sketch:

Steps:

(1) Prove for finite dimensional CW complexes $B = B^n$ ← meaning, prove theorem for all $F \to E \to B$ satisfying hypotheses where B is finite-dim'l CW.

(2) Prove for all CW complexes $B = \bigcup_{n \geq 0} B^n$ ← by 'finite-dim'l approximation' (we'll skip this)

(3) Prove for all spaces by "CW-approximation" theorem. ← (we'll skip this too).

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Re (1): For finite dim'l CW complexes, we'll induct on $\dim(B)$.

- true when B is 0-dim'l (b/c in this case $E = \coprod_{x \in B^0} \{F_x\}$)

  In this case $H^\bullet(B) = H^0(B^0) = \prod_{x \in B^0} \mathbb{Z}\langle 1_x \rangle$, and $H^k(E) = \prod_{x \in B^0} H^k(F_x) \underset{\text{check}}{\cong} H^k(F) \otimes H^0(B^0)$

  (exercise: spell out details)

- Say it's true for all $(n-1)$-dim'l CW complexes, and let

  $B = B^{(n-1)} \cup \bigcup_{\alpha \in A} e_n^\alpha$ (along $\varphi_\alpha^n: \partial e_n^\alpha \to B^{(n-1)}$).

Have $F \to E \to B$ satisfying hypotheses of L-H.  $B^{(n-1)}$

- Pick $x_\alpha \in \mathrm{int}(e_n^\alpha)$ for each α, and let $\mathring{e}_n^\alpha := e_n^\alpha \setminus x_\alpha$.

  Let $B' := B^{(n-1)} \cup \bigcup \mathring{e}_n^\alpha \subseteq B$, and denote by $E|_{B'} =: E'$

<u>First observation</u>: B' deformation retracts to $B^{(n-1)}$ (by retracting each $\mathring{e}_n^\alpha$ to ∂e_n^α),

and we want to similarly deduce that $E|_{B^{(n-1)}} \underset{\substack{\text{homotopy}\\ \text{equiv.}}}{\simeq} E|_{B'}$ (hence induces iso. on coh. groups)

apply below Lemma to $X = B'$, $X' = B^{(n-1)}$:

<u>Lem</u>: Given $\pi : P \to X$ (X paracompact) fiber bundle, say X def. retracts to $X' \subset X$. Then $P|_{X'} \subset P|_X$ is a homotopy equivalence.

(pf omitted in lecture, follows from applying relative version of the homotopy lifting property.)

(Pf <u>sketch</u>: Let $f_t : X \to X'$ be the def. retraction, i.e., $f_0 = id_X$, $f_1(X) \subset X'$, $f_t|_{X'} = id_{X'}$.

Look at

$$\begin{tikzcd} & & P \arrow[d, "\pi"] \\ P \arrow[rru, dashed, "id_P \text{ lifts } \pi"] \arrow[r, "\pi"'] \arrow[rr, bend right, "f_0 \circ \pi = id \circ \pi = \pi"'] & X \arrow[r, "f_0"'] & X \end{tikzcd}$$

restrict to this

lift at $t=0$

$$\begin{tikzcd} & & P \arrow[d, "\pi"] \\ P \arrow[rru, dashed, "\text{rel HLP: } \exists h_t"] \arrow[r, "\pi"'] \arrow[rr, bend right, "f_t \circ \pi"'] & X \arrow[r, "f_t"'] & X \end{tikzcd}$$

restricting to this lift along P'!

restricted to X':

$$\begin{tikzcd} & & P \arrow[d, "\pi"] \\ P' \arrow[rru, "id|_{P'} \text{ lifts } f_t|_{X'} \circ \pi = \pi"] \arrow[r, "\pi"'] & X' \arrow[r, "f_t|_{X'} = id_{X'}"'] & X \end{tikzcd}$$

By relative homotopy lifting property, if we denote by g_t the map $f_t \circ \pi : P \to X$, g_t admits a lift

$h_t : P \to P$ (i.e., $\pi \circ h_t = g_t = f_t \circ \pi$) agreeing w/ given lift id_P at time 0 and

w/ fixed lift $id_{P'}$ for all time when restricted to P'!

Check: h_t provides homotopy between id_P and $P \xrightarrow{h_1} P' \xrightarrow{incl} P$; ~~i.e.,~~ &

since $h_1|_{P'} = id|_{P'}$, i.e., $\begin{tikzcd} P' \arrow[r, "incl"] \arrow[rr, bend right] & P \arrow[r, "h_1"] & P \end{tikzcd}$, h_1 & incl. are htpy inverse. ∎ .)

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(R implicit)

Consider the following commutative diagram (using a fixed cohomology extension of the fibre) of LES's:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d, "\Phi"] & H^\bullet(B,B')\otimes H^\bullet(F) \arrow[r] \arrow[d, "\Phi\ (A)"] & H^\bullet(B)\otimes H^\bullet(F) \arrow[r] \arrow[d, "\Phi\ (?)"] & H^\bullet(B')\otimes H^\bullet(F) \arrow[r, "\delta^*"] \arrow[d, "\Phi\ (B)\ \checkmark"] & \; \arrow[d, "\Phi"] \\
\cdots \arrow[r] & H^\bullet(E,E') \arrow[r] & H^\bullet(E) \arrow[r] & H^\bullet(E') \arrow[r, "\delta^*"] & \;
\end{tikzcd}$$

relative version of same map Φ using relative cup product

$$\underset{\substack{\pi^*(\text{class} \\ \text{in } H^\bullet(B,B'))}}{H^\bullet(E,E')} \otimes_R \underset{c_j}{H^\bullet(E)} \xrightarrow{\smile} H^\bullet(E,E')$$

hypothesis of L-H.

(top seq. is exact b/c it was LES for pair (B,B') $\otimes$ a free module $H^\bullet(F)$).

(bottom seq. is LES of (E,E')).

exercise: check it's commutative. (Φ is natural, & check compat. w/ δ^* above)

If (A) & (B) are isomorphisms, then (?) will be too, by 5 lemma.

The map (B) is an iso. by induction, because:

$$\begin{tikzcd}
H^\bullet(B^{(n-1)})\otimes H^\bullet(F) \arrow[d, "\Phi"', "\text{by induction is}"] & H^\bullet(B')\otimes H^\bullet(F) \arrow[l, "\cong"'] \arrow[d, "\Phi \ \leftarrow \text{therefore this map is an} \cong"] \\
H^\bullet(E|_{B^{(n-1)}}) \arrow[ur, phantom, "\circlearrowleft"] & H^\bullet(E')\otimes H^\bullet(F) \arrow[l, "\cong"', "\text{(lemma above)}"]
\end{tikzcd}$$

Suffices to check (A) is an iso. By fibre bundle property, $\exists$ open U_α in $\mathrm{int}(e^n_\alpha)$ of x_α along which $E|_{U_\alpha} \cong F\times U_\alpha$ a trivial fibre bundle.

Let $U = \coprod_\alpha U_\alpha$, and let $U' = U\cap B'$ (i.e., $U' = U - \bigcup x_\alpha$). So $E|_U \cong F\times U$.

Excision $\Rightarrow$ $H^\bullet(B,B') \cong H^\bullet(U,U')\ (\cong H^\bullet(\coprod U_\alpha, \coprod(U_\alpha - x_\alpha)))$

and $H^\bullet(E,E') \cong H^\bullet(E|_U, E|_{U'}) \cong H^\bullet(U\times F, U'\times F)$.

Thus, (A) reduces to showing that

$$\Phi: H^\bullet(U,U')\otimes_R H^\bullet(F) \longrightarrow H^\bullet(U\times F, U'\times F) \text{ is an iso.}$$

using LES of the pair (U,U') & 5 lemma, it suffices to show for any V, the map

$\Phi : H^\bullet(V) \otimes H^\bullet(F) \longrightarrow H^\bullet(V \times F)$ is an iso. where Φ constructed using a coh. extension of fiber. (i.e., Leray – Hirsch for trivial bundles)

Exercise: Prove L-H for trivial bundles. i.e., $E = V \times F$, $H^k(F)$ free finitely gen $\forall k$ & let $c_j \in H^\bullet(E)$ be any collection of classes restricting to a basis $\{\gamma_j\}$ of $H^\bullet(F)$. Then prove that

$$H^\bullet(V) \otimes H^\bullet(F) \xrightarrow{\Phi} H^\bullet(E) \text{ is an iso.}$$

$$a \times \gamma_j \longmapsto \pi^*(a) \cup c_j \, .$$

(True by Künneth if one uses $\hat{c}_j = \pi_F^* \gamma_j$. Prove for a gen'l c_j by relating this choice to $\hat{c}_j$).

$\boxtimes$

Prop: (Splitting principle) (we'll state for cplx vec bundles, but real case analogous w/ 'same' proof).

Given any X (paracompact), any complex v.b. $E \to X$, $\exists$ a space Z and a map $s : Z \to X$ such that

(a) $s^*E \longrightarrow Z$ is isomorphic to a direct sum of line bundles.

(b) $s^* : H^*(X; \mathbb{Z}) \longrightarrow H^*(Z; \mathbb{Z})$ is injective.

(statement for real vector bundles involves injectivity of s^* on $H^*(-; \mathbb{Z}/2)$).

First, a quick observation: If $\begin{matrix} F \subset E \\ \downarrow \swarrow \\ X \end{matrix}$ vector sub-bundle of E, then using a fiberwise (Hermitian) metric structure

always exists if X paracompact. (by partition of unity argument)

(i.e., a 'continuous' family of $\langle -,- \rangle_p$ on E_p's) can define the orthogonal complement of a sub-bundle.

$F^\perp \subset E$ by $(F^\perp)_p := (F_p)^\perp$ using $\langle -,- \rangle_p$ on E_p.

depends on metric

This is a vector sub-bundle of E, complementary to F in each fiber $\Rightarrow$ get an iso. of vector bundles $E \cong F \oplus F^\perp \cong F \oplus E/F$, i.e., $E \cong F \oplus E/F$.

this bundle is defined w/o $\langle -,- \rangle_p$ but $\cong$ uses $\langle -,- \rangle_p$.

Pf of splitting principle:

By induction on $\mathrm{rank}_{\mathbb{C}}(E)$:

- true when $\mathrm{rank}_{\mathbb{C}}(E) = 1$. ✓.
- general case of rank k (assuming the for all rank $(k-1)$ vec. bundles on all paracompact spaces):

$E \to X$ rank k. Let $Z_1 = \mathbb{P}(E) \xrightarrow{s_1 = \pi} X$ (fibers are $\mathbb{CP}^{k-1}$'s)

fiberwise cplex projectivization.

Recall that Leray-Hirsch applies to $\mathbb{P}(E)$ using coh. extension of fiber given by $(1, h_p, \_\_\_)$

$\Rightarrow \pi^* = s_1^* : H^\bullet(X) \longrightarrow H^\bullet(\mathbb{P}(E))$ is injective.

(b/c $H^\bullet(\mathbb{P}(E))$ is freely gen. as a $H^\bullet(X)$-module (module str. comes from s_1^* & $\cup$) by 1, other cl's x's).

Looking at $\tilde{E} = s_1^* E \longrightarrow \mathbb{P}(E)$; the fiber at a point $(\overset{X}{\underset{\in}{x}}, \overset{E_x}{\underset{\cup\text{I}}{\ell}}) \in \mathbb{P}(E)$ is E_x.

In particular, the tautological line bundle $L_{taut} \to \mathbb{P}(E)$ is actually a vector sub-bundle of $\tilde{E}$:

$$L_{taut} \subseteq \tilde{E}$$

fiber over $(x, \ell \subseteq E_x)$ is ℓ $\quad \subseteq \quad$ fiber over $(x, \ell \subseteq E_x)$ is E_x

By observation right above the proof, paracompactness $\Rightarrow$ (using metric structure, e.g.,)

can split $\qquad \tilde{E} = L_{taut} \oplus \underbrace{E_1}$.

cplx. vector bundle over $Z_1 := \mathbb{P}(E)$ of rank $(k-1)$.

By inductive hypothesis, $\exists$ $s_2 : Z \to Z_1$ w/ s_2^* injective on cohomology

and $s_2^* E_1 \cong L_2 \oplus \cdots \oplus L_k$

$\Rightarrow s := s_1 \circ s_2 : Z \longrightarrow Z_1 \longrightarrow X$ satisfies:

- $s^* = s_2^* s_1^*$ injective on $H^\bullet(-;\mathbb{Z})$
- $s^* E = s_2^*(s_1^* E) = s_2^* \tilde{E} = s_2^*(L_{taut} \oplus E_1)$

$$\cong \underbrace{L_1}_{\substack{\shortparallel \\ s_2^* L_{taut}.}} \oplus L_2 \oplus \cdots \oplus L_k$$

⊠

Unwinding the induction, we can spell out what the final Z is:

$$\begin{tikzcd} Z \arrow[rrrr, bend left, "s"] & \cdots \arrow[r] & Z_2 \arrow[r] \arrow[d, phantom, "\shortparallel"] & Z_1 \arrow[r] \arrow[d, phantom, "\shortparallel"] & X \\ & & \mathbb{P}(L^{\perp}_{\text{taut}}) & \mathbb{P}(E) & \end{tikzcd}$$

in $s_1^*(E) = \tilde{E}$ over $\mathbb{P}(E)$. $\quad$ ($E_1 = L^{\perp}_{\text{taut}}$ for some metric on E)

Point in the fiber of Z_2 over $(x,\ell) \in \mathbb{P}(E) = Z_1$ is a line $L_2 \subseteq L_1^{\perp} = E_1 \subseteq E_x$.

Thus: If we use a fixed Hermitian metric on E (inducing one on all its pullbacks & sub-bundles)

$s: Z \to X$ has fiber over $x \in X$ equal to $\left\{ (L_1, \dots, L_k) \;\middle|\; \begin{array}{l} L_i \subseteq E_x \text{ line} \\ L_i \perp L_j \text{ for } i \neq j \text{ using } \langle -,- \rangle_x \\ \Rightarrow L_1 \oplus \cdots \oplus L_k = E_x \end{array} \right\}$

If V cplx. vec. space dimension n w/ inner product, the complex flag manifold

$$\mathbb{F}(V) = \{ (\ell_1, \dots, \ell_n) \mid \ell_i \perp \ell_j \} \qquad V_i := \ell_1 \oplus \cdots \oplus \ell_i.$$

w/o an inner product, can still describe as $\mathbb{F}(V) = \{ (V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n) \text{ w/ } \dim V_i = i \}$.

($\mathbb{F}(V)$ & $\mathbb{P}(V)$ sit within a wide collection of (generalized) flag manifolds)

Above, we see that $s: Z \to X$ is a fiber bundle w/ fiber $\mathbb{F}(E_x)$.

As mentioned, everything above works for real vec. bundles as well, using $Z :=$ real version of $\mathbb{F}(E) \to X$.

Some computations (starting with Stiefel-Whitney classes):

Smooth manifolds come equipped w/ a natural vector bundle, their tangent bundle $\begin{array}{c} TM \\ \downarrow \\ M^m \end{array}$ (real rank m vec. bundle; m = dimension). We can use the axioms to compute $w_i(TM) \underset{\text{shorthand}}{=:} \boxed{w_i(M)}$.

Ex: $S^n \subseteq \mathbb{R}^{n+1}$ unit sphere.

Recall that we can explicitly define $T_x S^n = \{ v \in \mathbb{R}^{n+1} \mid v \perp x \}$ (using $\langle -,- \rangle_{\text{Euclidean}}$)

In particular, $T_x S^n \subseteq \mathbb{R}^{n+1}$ inducing $TS^n \subseteq \underline{\mathbb{R}}^{n+1}$ ← trivial bundle over S^n, & moreover there's a direct sum decomposition

$$T_x S^n \oplus \mathbb{R} \xrightarrow{\cong} \mathbb{R}^{n+1}, \qquad (v, t) \longmapsto v + tx$$

(v orthogonal to x), inducing an iso. $TS^n \oplus \underline{\mathbb{R}} \xrightarrow{\cong} \underline{\mathbb{R}}^{n+1}$.

By Whitney Sum formula,

$$\Rightarrow w(TS^n) \cup \underbrace{w(\underline{\mathbb{R}})}_{1 = w_0} = \underbrace{w(\underline{\mathbb{R}}^{n+1})}_{1 = w_0}$$

$$\Rightarrow \underset{H^0}{1} + \underset{H^1(S^n;\mathbb{Z}/2)}{w_1(S^n)} + \underset{H^2}{w_2(S^n)} + \cdots = 1.$$

$\Rightarrow w_i(S^n)\left(\overset{\text{def}}{=} w_i(TS^n)\right) = 0$ for all $i > 0$. Note TS^n is not always trivial! (HW exercise, e.g., TS^{2k} has no non-vanishing sections).

(In genl, say a vector bundle E (rank k) is *stably trivial* if

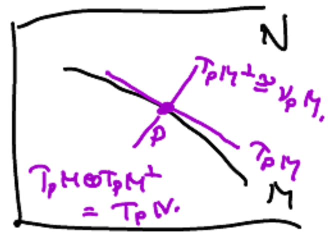
$$E \oplus \underline{\mathbb{R}}^{\ell} \cong \underline{\mathbb{R}}^{k+\ell} \quad \text{for some } \ell.$$

Whitney sum formula as above $\Rightarrow$ stably trivial E have $w_i(E) = 0$ for all i so w_i not a complete invariant).

In genl, we can study submanifolds $M^m \overset{\iota}{\subset} N^n$ via characteristic classes using the fact that

$$\underbrace{TN|_M}_{\text{same as } i^*TN \to M} \underset{\text{using a metric}}{\cong} \underset{\text{subbundle of } TN|_M}{TM} \oplus (TM)^\perp \cong TM \oplus \nu M$$

νM: normal bundle to $M \subseteq N$ $:= TN|_M \big/ TM$.



More on this next time.