—

Let's understand $T\,\underbrace{Gr_k(\mathbb{R}^n)}_{k\text{-planes in } \mathbb{R}^n}$. To understand Tangent bundle, first understand manifold structure.

Let $E_0 \in Gr_k(\mathbb{R}^n)$ any point (i.e., $E_0 \subseteq \mathbb{R}^n$ k-dim'l). So $\mathbb{R}^n = E_0 \oplus E_0^{\perp}$ ← using $\langle -,- \rangle_{Eucl.}$

consider the map

$$\psi_{E_0} : \operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp}) \longrightarrow Gr_k(\mathbb{R}^n)$$

$$a \longmapsto \text{graph of } a = (\underbrace{id \oplus a}_{E_0 \to E_0 \oplus E_0^{\perp}})(E_0) \underset{k\text{-dim'l}}{\subseteq} E_0 \oplus E_0^{\perp} \cong \mathbb{R}^n.$$

(b/c $id \oplus a$ injective)

Claim (exercise): Image of ψ_{E_0} is an open nhood of E_0, U_{E_0}; the maps $\psi_{E_0}^{-1} : U_{E_0} \longrightarrow \operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp}) \underset{\text{linearly}}{\cong} \mathbb{R}^{k(n-k)}$ makes $Gr_k(\mathbb{R}^n)$ into a smooth $k(n-k)$-dim'l manifold.

The tangent space at $E_0 \in Gr_k(\mathbb{R}^n)$ is isomorphic to $\operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp})$:

$$\star \quad d(\psi_{E_0})_0 : T_0 \operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp}) \longrightarrow T_{E_0} Gr_k(\mathbb{R}^n).$$
$$T_0 \operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp}) \cong \operatorname{Hom}_{\mathbb{R}}(E_0, E_0^{\perp})$$

Globalizing, let $E_{taut} \to Gr_k(\mathbb{R}^n)$ the tautological rank k vec. bundle $(E_{taut})_{E_0} = E_0$; we have

(fiber at E_0 is E_0) $\subseteq$ (fiber at E_0 is $\mathbb{R}^n$).

$$\begin{tikzcd} E_{taut} \arrow[r, phantom, "\subseteq"] \arrow[dr] & \underline{\mathbb{R}}^n \arrow[d] \\ & Gr_k(\mathbb{R}^n) \end{tikzcd}$$

Now using $\langle -,- \rangle_{Eucl}$ on $\underline{\mathbb{R}}^n$ we can split $\underline{\mathbb{R}}^n \cong E_{taut} \oplus E_{taut}^{\perp}$.

and there is an isomorphism of vector bundles $\underline{\operatorname{Hom}}(E_{taut}, E_{taut}^{\perp}) \xrightarrow{\cong} T\,Gr_k(\mathbb{R}^n)$ over $Gr_k(\mathbb{R}^n)$

$$(E_0, v) \longmapsto (E_0, \underbrace{d(\psi_{E_0})_0(v)}_{\star}).$$

$v \in \underline{\operatorname{Hom}}(E_{taut}, E_{taut}^{\perp})_{E_0} = \operatorname{Hom}(E_0, E_0^{\perp})$

(check: really a map of vector bundles, i.e., continuous).

Sub-example: $\mathbb{R}P^{n-1} = Gr(\mathbb{R}^n)$

$L = L_{taut}$ tautological line bundle. By above $T\mathbb{RP}^{n-1} = \underline{Hom}_{\mathbb{R}}(L, L^{\perp})$,

so $T\mathbb{RP}^{n-1} \oplus \underline{\mathbb{R}} \cong \underline{Hom}_{\mathbb{R}}(L, L^{\perp}) \oplus \underbrace{\underline{\mathbb{R}}}_{\cong}$

$L^* \otimes L \cong \underline{Hom}_{\mathbb{R}}(L, L)$ (works only for line bundles)

$$\cong \underline{Hom}_{\mathbb{R}}(L, L^{\perp} \oplus L)$$

$$\cong \underline{Hom}_{\mathbb{R}}(L, \underbrace{\underline{\mathbb{R}}^n}_{\cong \underbrace{\underline{\mathbb{R}} \oplus \cdots \oplus \underline{\mathbb{R}}}_{n}}) \cong \bigoplus_{i=1}^{n} \underline{Hom}_{\mathbb{R}}(L, \underline{\mathbb{R}}) \cong \underbrace{L^* \oplus \cdots \oplus L^*}_{n \text{ copies.}}$$

So, $T\mathbb{RP}^{n-1} \oplus \underline{\mathbb{R}} \cong \underbrace{L^* \oplus \cdots \oplus L^*}_{n \text{ copies.}}$

This implies $w(T\mathbb{RP}^{n-1}) = w(\underbrace{L^* \oplus \cdots \oplus L^*}_{n \text{ copies}})$ by Whitney sum formula. $\Big(w(T\mathbb{RP}^{n-1}) \cup \overbrace{w(\underline{\mathbb{R}})}^{1} = w(L^{*\oplus n}) \Big)$

$L \to \mathbb{RP}^{n-1}$ tautological line bundle then $L^* \otimes L \cong \underline{\mathbb{R}}$ implies that

$w_1(L^*) + w_1(L) = 0$ (b/c $w_1(L \otimes L') = w_1(L) + w_1(L')$ — we showed this earlier in class — for line bundles).

$\Rightarrow w_1(L^*) = -w_1(L) = w_1(L) = h.$

(as w_1 is defined on $H^1(\mathbb{RP}^{n-1}; \mathbb{Z}/2)$.).

So, $w(L^*) = 1 + h$, so Whitney sum formula implies: in $H^*(\mathbb{RP}^{n-1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[h]/h^n$, $|h| = 1$

$$w(\mathbb{RP}^{n-1}) := w(T\mathbb{RP}^{n-1}) = w((L^*)^{\oplus n}) = (1+h)^n$$

$$= 1 + \underset{H^1}{nh} + \binom{n}{2}\underset{H^2}{h^2} + \cdots + n h^{n-1} + \cancel{h^n}$$

$h^n = 0$ in

under the iso $H^i(\mathbb{RP}^{n-1}; \mathbb{Z}/2) \cong \mathbb{Z}/2$ sending $h^i \mapsto 1$,

$$\Rightarrow \boxed{w_i(\mathbb{RP}^{n-1}) = \binom{n}{i} \bmod 2}$$

i.e., $\boxed{w_i(\mathbb{RP}^n) = \binom{n+1}{i} \bmod 2.}$

Consequences:

Def: Say M^n is <u>parallelizable</u> if $TM \cong \underline{\mathbb{R}}^n \implies w(M) := w(TM) = 1$.

The computation above reveals that

Cor: $\mathbb{RP}^n$ can only possibly be parallelizable if $n = 2^k - 1$.

(Pf: unless $n = 2^k - 1$ some k, $\exists$ i with $\binom{n+1}{i}$ odd, hence that $w_i(T\mathbb{RP}^n) \neq 0$).

Suppose $\mathbb{R}^{q+1}$ admits <u>a bilinear product $\mathbb{R}^{q+1} \times \mathbb{R}^{q+1} \to \mathbb{R}^{q+1}$ w/o zero divisors</u>;

when is this possible? (e.g., possible for $q=1$, using complex mult. $\mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{C} \times \mathbb{C} \xrightarrow{\times} \mathbb{C} \cong \mathbb{R}^2$).

<u>Exercise</u>: can prove that if $\mathbb{R}^{q+1}$ has such a mult, then $T\mathbb{RP}^q$ has q linearly independent sections & is therefore trivial; i.e., $\mathbb{RP}^q$ must be parallelizable.

Cor: $\mathbb{R}^{q+1}$ can only admit such a product if $q = 2^k - 1$.

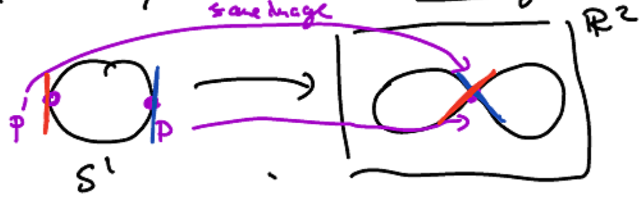
(in fact more strongly only have such a product when $q = 0, 1, 3, 7$, but these methods don't tell us that.)
reals complex quaternions octonions

<u>Immersions & embeddings</u>

If $f: M^m \to N^n$ smooth map w/ $df_x : T_xM \to T_xN$ is <u>injective</u> $\forall x \in M$,

say f is an <u>immersion</u> ($\Rightarrow \dim(N) \geq \dim(M)$). We say an immersion $f: M \to N$ is an <u>embedding</u> if further f is (proper) & injective.

not always required



immersion of $S^1 \to \mathbb{R}^2$ which is not an embedding

Special case of an embedding:

a submanifold $M \overset{i}{\subset} N$.

We can think of $\{df_x\}_x$ as inducing an (fiberwise) injective morphism of vector bundles:

$$\begin{array}{ccc} TM & \overset{df}{\hookrightarrow} & f^*TN \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

(i.e., $df_x : T_xM \to (f^*TN)_x = T_{f(x)}N$.)

If $f = i : M \hookrightarrow N$ inclusion, $i^*TN = TN|_M$.

For any immersion $f: M \to N$ (including embeddings) there is an associated <u>normal bundle</u>

← fiberwise quotient of vector bundles

$$\begin{array}{c}\nu_M\\ \downarrow\\ M\end{array} \quad \text{defined by} \quad \nu_M := \underbrace{f^*TN \big/ \underbrace{df(TM)}_{\text{sub-bundle of } f^*TN.}}_{} \qquad \left(\text{submanifold } M\subset N: \ \nu_M := \tfrac{TN|_M}{TM}\right).$$

$$\downarrow$$

$$M$$

ν_M is a vector bundle of rank $n-m$, a choice of metric induces an isomorphism

$$f^*TN \cong df(TM) \oplus df(TM)^\perp \qquad \left(\text{sub-manifold: } TN|_M \cong TM \oplus \nu_M\right).$$

$$\cong TM \oplus \nu_M.$$

This, plus Whitney sum formula, allows one to ~~understand~~ constrain/understand properties of embeddings & immersions provided one has control over TM, TN, (e.g., by telling us constraints on what char. classes of ν_M have to be).

Ex: $N = \underline{\mathbb{R}^n}$, so $TN = \underline{\mathbb{R}^n}$.

The Whitney sum formula tells us, for any immersion $M^m \overset{\text{immersion}}{\looparrowright} \mathbb{R}^n$ (could be an embedding)

since $\underset{\text{rank } m}{TM} \oplus \underset{\text{rank } n-m}{\nu_M} \cong \underline{\mathbb{R}^n}$

$$\Rightarrow \underbrace{w(TM)}^{w=1+w_1+\cdots} \cup \underbrace{w(\nu)}_{\bar{w}=1+\bar{w}_1+\bar{w}_2+\cdots} = 1. \leftarrow \text{sometimes called the "Whitney duality formula"}$$

(up to stabilizing, ν_M is "dual" or "inverse" via $\oplus$ to TM).

Can solve for $w(\nu)$ as $w(TM)$ is a unit.

in deg 1: $w_1 + \bar{w}_1 = 0 \Rightarrow \underline{\bar{w}_1 = -w_1} = w_1 \pmod 2.$

in deg 2: $w_2 + w_1\bar{w}_1 + \bar{w}_2 = 0$

using deg 1 solution of $\bar{w}_1$

$w_2 + w_1^2 + \bar{w}_2 = 0 \Rightarrow \underline{\bar{w}_2 = w_2 + w_1^2}.$

etc.

For any M, let $\bar{w}(M)$ be the solution to $w(M) \cup \bar{w}(M) = 1$ (know: $w(\nu_M) = \bar{w}(M)$ for any $M \looparrowright \mathbb{R}^n$).

e.g., $w(\mathbb{RP}^m) = \underline{(1+h)^{m+1}}$ in $\mathbb{Z}/2[h]/h^{m+1} \cong H^*(\mathbb{RP}^m; \mathbb{Z}/2)$

so $\bar{w}(\mathbb{RP}^m)$ is $\dfrac{1}{(1+h)^{m+1}}$ in

Let's explicitly compute in some nice cases:

identity: $(1+h)^2 = 1+h^2$ over $\mathbb{Z}/2$, similarly $(1+h)^{2^i} = 1+h^{2^i}$ mod 2, so

if $m+1 = \sum n_i 2^i$ binary representation of $m+1$,

then over $\mathbb{Z}/2$, $(1+h)^{m+1} = (1+h)^{\sum n_i 2^i} = \prod_{i \text{ s.t. } n_i=1} (1+h^{2^i})$

e.g., $m=10$: $(\mathbb{RP}^{10})$.

$w(\mathbb{RP}^{10}) = (1+h)^{11=1+2+8} = (1+h)(1+h^2)(1+h^8) = 1+h+h^2+h^3+h^8+h^9+h^{10}$.

in $\mathbb{Z}/2[h]/h^{11}$ $\qquad$ in $\mathbb{Z}/2[h]/h^{11}$.

To compute $\bar{w}$ in this case (mult. inverse of $(1+h)^{m+1}$ in $\mathbb{Z}/2[h]/h^{m+1}$), observe:

for any s w/ $2^s > m$,

$$\underbrace{(1+h)^{m+1}}_{w}(1+h)^{2^s-(m+1)} = (1+h)^{2^s} \underset{(\text{mod } 2)}{=} 1+h^{2^s} = 1 \quad (h^{2^s} = 0 \text{ in }).$$

$\Rightarrow \bar{w} = (1+h)^{2^s-(m+1)}$ for any such s.

$m=10$ again; e.g.,

$\bar{w} = (1+h)^{16-(11)} = (1+h)^{5=1+4} = (1+h)(1+h^4) = 1+h+h^4+h^5$.

i.e., $\bar{w}_5 = h^5 \neq 0$. (implies: if $\mathbb{RP}^{10} \hookrightarrow \mathbb{R}^n$, then $\bar{w}_5(\mathbb{RP}^{10}) = w_5(\nu_M) \neq 0$ so rank $(\nu_M) = n-10 \geq 5$ by dimension reasons).

<u>Cor</u>: If $\mathbb{RP}^{10} \hookrightarrow \mathbb{R}^n$ then $n \geq 15$.

i.e., $\mathbb{RP}^{10}$ can't immerse or embed into $\mathbb{R}^{14}$.

(we know by whitney embedding any $M^m \hookrightarrow \mathbb{R}^{2m+1}$, but often can do better, e.g., $S^2 \hookrightarrow \mathbb{R}^3$, the above cor puts constraints on how much better one can do for case of $\mathbb{RP}^{10}$).

In general, the amount of "immersion constraint" we'll get for a given $\mathbb{RP}^m$ depends on m. One case in which it's very strong:

$\mathbb{RP}^{2^k}$: Get $w(\mathbb{RP}^{2^k}) = (1+h)^{2^k+1}$, and $\bar{w}(\mathbb{RP}^{2^k}) = (1+h)^{\overbrace{2^{k+1}}^{2^s \text{ from above (chose } s=k+1)}-2^k-1} = (1+h)^{2^k-1}$.

$$= \frac{(1+h)^{2^k}}{1+h} = \frac{1+h^{2^k}}{1+h} \underset{\text{mod } 2}{=} \underbrace{1+h+h^2+\cdots+h^{2^k-1}}.$$

Seeing as $\overline{w}_{2^k-1}(T\mathbb{RP}^{2^k}) \neq 0 \Rightarrow$ The normal bundle of any $\dim$ $\mathbb{RP}^{2^k} \hookrightarrow \mathbb{R}^n$ $\underbrace{\text{must have dimension}}_{n-2^k} \geq 2^k - 1$, i.e., $n \geq 2(2^k) - 1$.

Cor: For $m = 2^k$, $\mathbb{RP}^m$ can't immerse (hence can't embed either) into $\mathbb{R}^{2m-2}$

(Whitney's immersion theorem states any $M^m \hookrightarrow \mathbb{R}^{2m-1}$, & Cor. states that for $\mathbb{RP}^m$, $m = 2^k$ we can't $\hookrightarrow$ into anything lower).

Stiefel-Whitney numbers (not covered in lecture)

X^n compact smooth manifold, . $w_i(X) := w_i(TX) = w_i \in H^i(X; \mathbb{Z}/2)$,

Can multiply $\prod w_i(X)^{n_i} \in H^{\sum i n_i}(X; \mathbb{Z}/2)$.

Recall that $\exists$ a canonical $\mathbb{Z}/2$ fundamental class $[X] \in H_n(X; \mathbb{Z}/2)$ (don't need orientability mod 2)

determining a morphism $H^n(X; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ (iso. if X connected)

$$\phi \longmapsto \langle \phi, [x] \rangle. \text{ or } \phi([X]).$$

$\Rightarrow$ Whenever $\sum i n_i = n$ we get a number,

denoted $\prod w_i^{n_i}[X] \in \mathbb{Z}/2$ by $\langle \prod w_i(X)^{n_i}, [X] \rangle$.

Stiefel-Whitney # of X.

0 in $\mathbb{Z}_2[h]/h^5$.

E.g., if $X = \mathbb{RP}^4$, $w(X) = (1+h)^{5=1+4} = (1+h)(1+h^4) = 1 + \underset{w_1}{h} + \underset{w_4}{h^4} + \cancel{h^5}$

the possible numbers here are:

$w_1^4[X]$

$w_2 w_1^2[X]$

$w_2^2[X]$

$w_4[X]$

but not all are non-zero, i.e., $w_2(\mathbb{RP}^4) = 0$ so $w_2 w_1^2[\mathbb{RP}^4] = 0$, & e.g., $w_4[X] = 1$.

e.g., a cobordism from $\emptyset$ to X is a W w/ $\partial W = X$.

In particular ($X_0 = \emptyset$), if $X = \partial W$ then all Stiefel-Whitney #'s of X are 0.

Prop: If X_0 & X_1 are cobordant they have the same Stiefel-Whitney #'s.

(Thom proved converse is true also, but that's much harder)

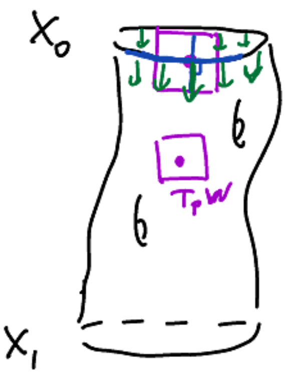
Cor: $\mathbb{RP}^4$ and S^4 are not cobordant. (we computed above that $w_4[\mathbb{RP}^4] = 1$ but $w_4[S^4] = 0$ b/c $w_4(S^4) = 0$ by last class).

special case of above

Cor: $\mathbb{RP}^4$ is not ∂W for a cpct m'fold-w/-∂ W.

Pf: Write $w = \prod w_i^{n_i}$ where $\sum i n_i = n = \dim X_i$. $i = 0,1$. $\left(w(E) = \prod w_i^{n_i}(E). \right)$

The basic observation is that for a cobordism W:



(always exists by partition of unity argument - Math 535a)

A choice of inward pointing vector field along $TW|_{X_i}$ leads to a decomposition $\underline{TW|_{X_i} \cong TX_i \oplus \underline{\mathbb{R}}}$

Therefore

$$w(TW)\Big|_{X_i} = w(TW|_{X_i}) = w(TX_i \oplus \underline{\mathbb{R}}) \underset{\text{Whitney sum}}{=} w(TX_i)$$

means pull back along $X_i \hookrightarrow W$

Furthermore, if $[X_i] \in H_n(X_i; \mathbb{Z}/2)$ denotes the canonical $\mathbb{Z}/2$ fund. classes, and $[W] \in H_{n+1}(W, \partial W; \mathbb{Z}/2)$ denotes the canonical rel. $\mathbb{Z}/2$ fund. class of W.

we know (or at least previously asserted) that in LES of pair $(W, \partial W)$ w/ $\mathbb{Z}/2$-coeffs,

$$H_{n+1}(W, \partial W) \xrightarrow{\partial_*} H_n(\partial W) \xrightarrow{i_*} H_n(W) \qquad \partial_* : [W] \longmapsto [\partial W].$$
$$[W] \longmapsto [\partial W]$$

Cor: if $i: \partial W \to W$ then $i_*[\partial W] = 0$ in $H_n(W)$.

$$i_*[\partial W] = i_*[X_0] + i_*[X_1] \qquad (\text{as } \partial W = X_0 \amalg X_1).$$

$\Rightarrow$ (mod 2) $\quad i_*[X_0] = i_*[X_1]$

Therefore if $w = \prod w_i^{n_i}$

$$\Rightarrow \langle w(TW), i_*[X_0] \rangle = \langle w(TW), i_*[X_1] \rangle \overset{\text{naturality}}{=} \langle w(TW|_{X_1}), [X_1] \rangle$$

$\| $ by naturality $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$ before

$\langle w(TW|_{X_0}), [X_0] \rangle \qquad\qquad\qquad\qquad\qquad\qquad \langle w(TX_1), [X_1] \rangle$

$\|$ before $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$$\langle w(JX_0), [X_0] \rangle$$

$$\shortparallel$$

$$\prod w_i^{n_i} [X_0].$$

$$\shortparallel$$

$$\prod w_i^{n_i} [X_1]$$

$\boxtimes$.