Some computations of Chern classes and Chern numbers.

One source of complex vector bundles comes from the tangent bundle to a complex manifold, as we'll now explain.

Some (fiber-wise) linear algebra.

V real vec. space of dim. $2n$. A cmplex str. on V is $J: V \to V$ w/ $J^2 = -id$.

Using J, V inherits str. of a $\mathbb{C}$-n-dim'l vec. space via $(a+bi)(v) := (a+bJ)(v)$, call this cplx. vector space (V, J).

Given a real vector bundle $E \to X$ of rank $2n$, a $J \in \Gamma(End(E))$ (i.e., $J_x: E_x \to E_x$) w/ $J^2 = -id$ (meaning $J_x^2 = -id$ for all x) induces a complex vec. bundle str. on E, call it (E, J) and hence Chern classes

Call such a J a (fiberwise) complex structure on E.

Call a pair (X, J) an almost complex manifold & such a J on TX an almost complex structure.

manifold fiberwise complex structure on TX

An almost cplx manifold (X, J) has $c_j(X) := c_j(TX, J)$.

A complex manifold (dim. n) is a space equipped w/ (equiv. class of, or maximal) atlas $\{\mathcal{U}_\alpha, \phi_\alpha : \mathcal{U}_\alpha \to \phi_\alpha(\mathcal{U}_\alpha) \underset{open}{\subset} \mathbb{C}^n\}_\alpha$, ($\mathcal{U}_\alpha \subset X$) whose transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : \underset{\overset{\cap \text{ open}}{\mathbb{C}^n}}{\phi_\alpha(\mathcal{U}_\alpha)} \longrightarrow \underset{\overset{\cap \text{ open}}{\mathbb{C}^n}}{\phi_\beta(\mathcal{U}_\beta)} \quad \text{are holomorphic,}$$

meaning that $d(\phi_\beta \circ \phi_\alpha^{-1}) \circ i = i \circ d(\phi_\beta \circ \phi_\alpha^{-1})$.

Lem: Any complex manifold X has a canonical almost complex structure, hence TX is a cplx. vec. bdle (& has Chern classes).

Sketch: At a given $p \in X$, pick a chart $\mathcal{U}_\alpha$ around p, giving

$$T_pX \xrightarrow[(d\phi_\alpha)_p]{\cong} T_{\phi_\alpha(p)}(\phi_\alpha(\mathcal{U}_\alpha)) \cong \mathbb{C}^n \circlearrowleft i$$

Define J_p to be $(dx)_p^{-1}\, i\, (d\phi_\alpha)_p$; check independent of choice & smoothly varying. (uses holomorphicity of transition functions). $\boxtimes$.

Ex: $G_k(\mathbb{C}^n)$. We can construct a complex differentiable atlas parallel to the (real) atlas we constructed for $G_k(\mathbb{R}^n)$.

(i.e., around $E_0 \in G_k(\mathbb{C}^n)$, obtain (inverse to) a chart map by

(using the standard Hermitian inner prod. on $\mathbb{C}^n$.

$$\Psi : \underset{\cong\, \mathbb{C}^{k(n-k)}}{\mathrm{Hom}_{\mathbb{C}}(E_0, E_0^{\perp})} \longrightarrow G_k(\mathbb{C}^n)$$

$$a \longmapsto \mathrm{graph}(a)(E_0). \quad)$$

$$\underset{\underset{(id,a)}{\parallel}}{\mathrm{graph}(a)} : E_0 \hookrightarrow E_0 \oplus E_0^{\perp} \cong \mathbb{C}^n$$

(exercise: complex manifold)

The same analysis previously applied to $G_k(\mathbb{R}^n)$ implies that as complex vector bundles

$$T\,Gr_k(\mathbb{C}^n) \cong \underline{\mathrm{Hom}}_{\mathbb{C}}(E, E^{\perp})$$

tautological bundle over $G_k(\mathbb{C}^n)$

inside $\underline{\mathbb{C}}^n$ using $\langle -,- \rangle_{\text{Euclidean}}$ Hermitian metric.

$\underline{k=1}$ $\quad (G_1(\mathbb{C}^n) \cong \mathbb{C}P^{n-1})$

$$\Rightarrow T\mathbb{C}P^{n-1} \oplus \underline{\mathbb{C}} \underset{\text{as before}}{\cong} \underline{\mathrm{Hom}}_{\mathbb{C}}(L_{taut}, L_{taut}^{\perp}) \oplus \underset{\underset{\underset{\underset{\mathbb{C}.}{\cong}}{L_{taut}^* \otimes L_{taut}}}{\cong}}{\mathrm{Hom}(L_{taut}, L_{taut})}$$

$$= \underline{\mathrm{Hom}}(L_{taut}, \underline{\mathbb{C}}^n) = \underbrace{L_{taut}^* \oplus \cdots \oplus L_{taut}^*}_{n \text{ times}}$$

Now for any cplx line bundle $L \to B$, $\underline{c_1(L^*) = -c_1(L)}$.

$$\Big(b/c \ \underset{\underset{c_1(L)+c_1(L^*)}{\parallel}}{c_1(L \otimes L^*)} = c_1(\underline{\mathbb{C}}) = 0 \Big).$$

h canonical generator in $H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$.

$$\text{So, } c_1(L_{taut}^*) = -c_1(L_{taut}) = -(-h) = h.$$

$$\text{so, } c(\mathbb{C}P^{n-1}) := c(T\mathbb{C}P^{n-1}) \underset{\text{whitney sum}}{=} c\big((L_{taut}^*)^{\oplus n}\big) \underset{\text{whitney sum}}{=} \prod_{i=1}^{n} \overset{\overset{(1+h)}{\parallel}}{c(L_{taut}^*)}$$

$$= (1+h)^n \quad \text{in } \underset{\underset{\mathbb{Z}[h]/h^n.}{\cong}}{H^{\bullet}(\mathbb{C}P^{n-1};\mathbb{Z})}$$

$$1 + nh + \binom{n}{2}h^2 + \cdots + nh^{n-1}.$$

$$\text{i.e., } c_i(\mathbb{C}P^{n-1}) = \binom{n}{i} h^i \in \underbrace{H^{2i}(\mathbb{C}P^{n-1};\mathbb{Z})}_{\underset{\mathbb{Z}\langle h^i \rangle.}{\cong}}.$$

Above it was convenient to know relationship between c_1's for L, L^*. What about E vs. E^*? (& all c_i's)

Lemma: E rank k complex vector bundle, and let $E^* := \underline{\mathrm{Hom}}_{\mathbb{C}}(E, \mathbb{C})$.
Then for each i, $c_i(E^*) \cong (-1)^i c_i(E)$.

Pf: • true when rank(E) = 1, by above. $\big(c_1(L^*) = -c_1(L),\ c_i(L^*) = 0 = (-1)^i c_i(L)$ for $i > 1$, & $c_0(L^*) = 1 = c_0(L)\big)$.

• true when $E \cong L_1 \oplus \cdots \oplus L_k$. $(\Rightarrow E^* \cong L_1^* \oplus \cdots \oplus L_k^*)$.

$$\Rightarrow \quad c(E^*) = \prod_{i=1}^{k} c(L_i^*) = \prod_{i=1}^{k} (1 - c_1(L_i))$$

vs. $c(E) = \prod_{i=1}^{k} (1 + c_1(L_i))$; now check in deg 2i: these differ by $(-1)^i$.

• In general, by splitting principle, ∃ $s : Z \longrightarrow \overset{\substack{E \\ \downarrow}}{X}$ w/ $s^*E \cong L_1 \oplus \cdots \oplus L_k$.

So it follows from previous case that

$$\begin{tikzcd}
(-1)^i c_i(s^*E) \arrow[r, phantom, "="] \arrow[d, phantom, "\parallel"] & c_i((s^*E)^*) \arrow[d, phantom, "\parallel"] \\
s^*((-1)^i c_i(E)) \arrow[r, dashed, equal] & s^*(c_i(E^*))
\end{tikzcd}$$

s^* is injective so $(-1)^i c_i(E) = c_i(E^*)$ ✓. $\boxed{2}$.

V vector space / ℂ, have $\overline{(-)} : \underset{a+ib \,\mapsto\, a-ib}{\mathbb{C} \xrightarrow{\cong} \mathbb{C}}$ real-linear involution; pulling back the action of ℂ on V by $\overline{(-)}$ gives a new complex vector space $\overline{V}$; as real vector spaces $V_{\mathbb{R}} = (\overline{V})_{\mathbb{R}}$, but

$$(a+bi) \underset{\substack{\uparrow \\ \text{scalar mult. in } \overline{V}}}{\cdot} v := (a-ib) \underset{\substack{\uparrow \\ \text{scalar mult. in } V.}}{\cdot} v$$

Observe that a Hermitian inner product $\langle -, - \rangle$ on V induces an ℂ-isomorphism $V^* \cong \overline{V}$.

complex linear in this factor

complex antilinear in this factor, meaning linear when thought of as a map from $\overline{V}$.

Similarly, from a complex vector bundle $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$, can construct $\begin{matrix} \overline{E} \\ \downarrow \\ X \end{matrix}$, & a choice of (fiberwise) Hermitian metric gives an iso. $\overline{E} \cong E^*$

Cor: $c_i(\overline{E}) = (-1)^i c_i(E)$.

Linear algebra of complexifications

V real vec. space dim n.

$\rotatebox{90}{$\leadsto$}$

$V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C}$ complexification.. complex vec. space of dim n.

observe: in contrast to arbitrary complex vec. space, V comes equipped w/ a canonical conjugation action:

$\mathbb{C} \xrightarrow{\overline{(-)}} \mathbb{C}$ induces (by $V \otimes_\mathbb{R} -$) $V_\mathbb{C} \xrightarrow[\cong]{\overline{(-)}} V_\mathbb{C}$ complex anti-linear isomorphism, i.e.,

induces a complex-linear isomorphism $V_\mathbb{C} \xrightarrow{\cong} \overline{V_\mathbb{C}}$.

Can recover V as $\mathrm{Fix}(V_\mathbb{C} \circlearrowleft \overline{(-)})$ (i.e., $+1$-eigenspace: note $\overline{(-)}^2 = \mathrm{id}$)

If W is a complex vector space, denote by $W_\mathbb{R}$ the underlying real vector space ($\dim_\mathbb{R} 2n$).
($\dim_\mathbb{C} = n$)

scalar mult. in W

Multiplication by i on W $\longleftrightarrow$ $J : W_\mathbb{R} \xrightarrow{w \longmapsto iw} W_\mathbb{R}$ w/ $J^2 = -\mathrm{id}$ ("complex structure on $W_\mathbb{R}$")

Lem: If $\overset{\dim_\mathbb{C} n}{W}$ complex vector space then $(\overset{\dim_\mathbb{R} 2n}{W_\mathbb{R}})_\mathbb{C} := \overbrace{(W_\mathbb{R} \otimes_\mathbb{R} \mathbb{C})}^{\dim_\mathbb{C} 2n} \cong W \oplus \overline{W}$.

Pf sketch: mult. by i on W induces as above $J : W_\mathbb{R} \to W_\mathbb{R}$ w/ $J^2 = -\mathrm{id}$.

$\Rightarrow$ get $J_\mathbb{C} = J \otimes_\mathbb{R} \mathrm{id}_\mathbb{C} : (W_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) \to (W_\mathbb{R} \otimes_\mathbb{R} \mathbb{C})$ w/ $(J_\mathbb{C})^2 = -\mathrm{id}$,

i.e., $J_\mathbb{C}$ has $(+i)$ and $(-i)$ eigenspaces, which together give a decomposition $(W_\mathbb{R})_\mathbb{C}$.

i.e., $W_\mathbb{R} \otimes \mathbb{C} \underset{\text{as } \mathbb{C}\text{-vec. spaces}}{\cong} W^+ \oplus W^-$ $\quad W^\pm := \pm i$ eigenspace.

So need to show $W^+ \cong W$ (& $W^- \cong \overline{W}$, & $\overline{(-)}$ on $(W_\mathbb{R})_\mathbb{C}$ swaps W & $\overline{W}$ factors). — exercise

Define $W \xrightarrow{T} W^+$; on the level of real vector spaces $W_\mathbb{R} \overset{v \mapsto v \otimes 1}{\hookrightarrow} (W_\mathbb{R})_\mathbb{C} \xrightarrow{\mathrm{pr}_{+i}} W^+$

which all together sends $w \overset{T}{\longmapsto} \frac{1}{2}(w \otimes 1 - Jw \otimes i)$. $\qquad \alpha \longmapsto \frac{1}{2}(\alpha - i J_\mathbb{C} \alpha)$

check: $Jw \overset{T}{\longmapsto} i(Tw)$; in particular.

$\alpha = (\alpha^+, \alpha^-) \in W^+ \oplus W^-$,

$iJ\alpha = (-\alpha^+, +\alpha^-)$

$\frac{1}{2}(\alpha - iJ\alpha) = (\alpha^+, 0)$.

T is a complex-linear map $W \to W^+$, isomorphism (check).

$\boxtimes$.

Pontryagin classes of real vector bundles

$E \to X$ real vec. bundle of rank k.

Form $E \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow X$ (fiberwise) complexification, complex rank k vec. bundle w/ an iso.

$$E \otimes_{\mathbb{R}} \mathbb{C} \underset{\text{conjugation}}{\cong} \overline{E \otimes_{\mathbb{R}} \mathbb{C}} \underset{\substack{\text{using} \\ \text{Hermitian metric} \\ \text{as in last time}}}{\cong} (E \otimes_{\mathbb{R}} \mathbb{C})^* \,. \quad (\bigstar)$$

Taking Chern classes $c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{2i}(X;\mathbb{Z})$, and $(\bigstar)$ implies:

$$\underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})} = c_i((E \otimes_{\mathbb{R}} \mathbb{C})^*) \underset{\text{last time}}{=} (-1)^i \, \underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})}.$$

If i is odd, this tells us that $2c_i(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$ in $H^{4k+2}(X;\mathbb{Z})$.
$i = 2k+1$

Def: $E \to X$ real vec. bundle of rank k, define its k^{th} Pontryagin class by

$$p_k(E) := (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(X;\mathbb{Z}).$$

By definition, $p_k(E) = 0$ if $2k > \text{rank}(E)$.

(to be continued next time)