Last time:

Pontryagin classes of real vector bundles

$E \to X$ real vec. bundle of rank k.

Form $E \otimes_{\mathbb{R}} \mathbb{C} \to X$ (fiberwise) complexification, complex rank k vec. bundle w/ an

iso. $E \otimes_{\mathbb{R}} \mathbb{C} \underset{\text{conjugation}}{\cong} \overline{E \otimes_{\mathbb{R}} \mathbb{C}} \underset{\substack{\text{using} \\ \text{Hermitian metric} \\ \text{as in last time}}}{\cong} (E \otimes_{\mathbb{R}} \mathbb{C})^*$. $(\star)$

Taking Chern classes $c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{2i}(X;\mathbb{Z})$, and $(\star)$ implies.

$$\underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})} = c_i((E \otimes_{\mathbb{R}} \mathbb{C})^*) \underset{\text{last time}}{=} (-1)^i \underline{c_i(E \otimes_{\mathbb{R}} \mathbb{C})}.$$

If i is odd, this tells us that $2c_i(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$ in $H^{4k+2}(X;\mathbb{Z})$.
$i = 2k+1$

Def: $E \to X$ real vec. bundle of rank n, define its k^{th} Pontryagin class by

$$p_k(E) := (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(X;\mathbb{Z}).$$

By definition, $p_k(E) = 0$ if $2k > \operatorname{rank}(E)$.

Whitney sum formula E, E' two vector bundles, then

$$p_k(E \oplus E') := (-1)^k c_{2k}((E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (E' \otimes_{\mathbb{R}} \mathbb{C}))$$

(Whitney sum for Chern classes)
$$= (-1)^k \underbrace{\sum_{\substack{i+j=2k \\ i \geq 0 \\ j \geq 0}} c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \smile c_j(E' \otimes \mathbb{C})}$$ (convention $c_0 = 1$)

terms where both i, j even

terms where one of i or j is odd.

$$= \sum_{r+s=k} (-1)^{k=r+s} c_{2r}(E \otimes \mathbb{C}) \smile c_{2s}(E \otimes \mathbb{C}) + (2\text{-torsion terms})$$

$$= \sum_{\substack{r+s=k \\ r\geq 0,\ s\geq 0}} p_r(E) \smile p_s(E') \quad + (2\text{-torsion terms}).$$

$\leftarrow$ convention that $p_0 = 1$.

So denoting $p(E) := \underbrace{1}_{= p_0(E)} + p_1(E) + p_2(E) + \cdots$ total Pontryagin class,

get
$$p(E \oplus E') = p(E)p(E') + 2\text{-torsion terms}.$$

Special case:

Say $E = F_{\mathbb{R}}$ rank $2n$ real vec. bundle for $F \to X$ a complex rank n vec. bundle.

Then, a fiberwise version of the lemma at the start of lecture implies:
$$(F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \cong F \oplus \overline{F} \underset{\substack{\text{Hermitian} \\ \text{metric } T}}{\cong} F \oplus F^*.$$

So, $\boxed{p_k(F_{\mathbb{R}})} = (-1)^k c_{2k}(F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(F \oplus F^*)$

$$\underset{\substack{\text{Whitney sum} \\ \text{for Chern classes}}}{=} (-1)^k \sum_{\substack{i+j=2k \\ i\geq 0\ j\geq 0}} c_i(F) \smile c_j(F^*) = \boxed{(-1)^k \sum_{\substack{i+j=2k \\ i\geq 0\ j\geq 0}} (-1)^j c_i(F) \smile c_j(F)}$$

As usual if Q a (real) smooth manifold denote $p_k(Q) := p_k(TQ)$.

Example: Compute $p_k(\mathbb{CP}^n)$.

We previously computed as complex vector bundles, $T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1 \text{ copies}}$ (☆) $L :=$ taut. line bundle

$$\Rightarrow c(T\mathbb{CP}^n) = (\underbrace{1+h}_{c(L^*)})^{n+1} \quad \text{in } H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}.$$

complex conjugating (☆), we get:

$$\overline{T\mathbb{CP}^n \oplus \underline{\mathbb{C}}} \cong \overline{\underbrace{L^* \oplus \cdots \oplus L^*}_{n+1}} \cong \underbrace{L \oplus \cdots \oplus L}_{n+1}$$

$$\overline{T\mathbb{CP}^n} \oplus \underline{\mathbb{C}} \quad (\overline{\mathbb{C}} \cong \mathbb{C}).$$

$$\Rightarrow c(\overline{T\mathbb{CP}^n}) = c(L)^{n+1} = (1-h)^{n+1}, \text{ in same ring}$$

So, $p_k(\mathbb{C}P^n) = p_k(T\mathbb{C}P^n) = (-1)^k c_{2k}(T\mathbb{C}P^n \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^k c_{2k}(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n})$.

$= (-1)^k \cdot (\deg 2k \text{ part of } (1+h)^{n+1}(1-h)^{n+1})$.

So, $p(\mathbb{C}P^n) = \sum_{k \geq 0} (-1)^k \left((1+h)^{n+1}(1-h)^{n+1} \right)_{\deg 2k \text{ part}}$

$= \sum_{k \geq 0} (-1)^k \left(\underbrace{(1-h^2)^{n+1}} \right)_{\deg 2k \text{ part}}$

deg $2k$ part is $(-1)^k \cdot$ deg $2k$ part of $(1+h^2)^{n+1}$, & no odd degree parts of this expression, hence.

$\boxed{= (1+h^2)^{n+1}}$ (again in $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[h]/h^{n+1}$).

Special case: $n = 2m$ is even. We get $p(\mathbb{C}P^{2m}) = (1+h^2)^{2m+1}$.

In particular $p_m(\mathbb{C}P^{2m}) := p_m(T\mathbb{C}P^{2m}) = \binom{2m+1}{m} h^{2m} \in H^{4m}(\mathbb{C}P^{2m};\mathbb{Z}) \cong \mathbb{Z}\langle h^{2m} \rangle$

top degree cohomology ($\dim_{\mathbb{R}} \mathbb{C}P^{2m} = 4m$).

Pairing w/ the fundamental class $[\mathbb{C}P^{2m}]$ (using complex orientation) sends $h^{2m} \longmapsto +1$, hence we get

$$\underbrace{\langle p_m(\mathbb{C}P^{2m}), [\mathbb{C}P^{2m}] \rangle}_{\text{the Pontryagin number } p_m[\mathbb{C}P^{2m}]} = \binom{2m+1}{m}$$

More generally, if X compact oriented manifold, for any collection $\overset{I}{\{n_i \geq 0\}}$ with $\sum 4 i n_i = \dim X$, can define

$$p_I[X] = \prod p_i^{n_i}[X] := \langle \underbrace{\prod p_i(TX)^{n_i}}_{\in H^{\dim X}(X;\mathbb{Z}) \text{ by hypothesis}}, [X] \rangle \in \mathbb{Z}.$$

Pontryagin numbers.

Observe: If $X \xrightarrow[f]{\cong} Y$ oriented diffeo. (so $f_*[X] = [Y]$) then naturality $\Rightarrow$

$\prod p_i^{n_i}[X] = \prod p_i^{n_i}[Y]$.

On the other hand, $\prod p_i^{n_i}[\overline{X}] = -\prod p_i^{n_i}[X]$.

means X w/ opposite orientation, $-[X]$.

(didn't say this in class)

Cor: If a single Pontryagin # is non-zero, then $X \underset{\text{oriented diff.}}{\not\cong} \overline{X}$.

Cor: $\mathbb{CP}^{2m} \not\cong \overline{\mathbb{CP}^{2m}}$.

Interestingly enough, $\mathbb{CP}^{2m+1} \underset{\text{oriented}}{\cong} \overline{\mathbb{CP}^{2m+1}}$. e.g., $\mathbb{CP}^1 = S^2 \xrightarrow{\text{reflection}} S^2 = \overline{\mathbb{CP}^1}$.)

Rmk: Also have numerical invariants of cpct not(necessarily) oriented manifolds, coming from Stiefel-Whitney numbers: For X a cpct manifold, and any $I = \{n_i \geq 0\}$ s.t. $\sum_i i n_i = \dim X$, get $w_I[X] := \langle \prod_i w_i^{n_i}(TX), [X] \rangle \in \mathbb{Z}/2$.
$w_I[X] = \prod w_i^{n_i}[X]$

It turns out that

- Stiefel-Whitney numbers are invariants of X up to cobordism, i.e., if $X \sim_{cob} X'$ meaning $\exists$ cpct $W^{\dim X+1}$ with $\partial W = X \amalg X'$, then $w_I[X] = w_I[X']$.

- Similarly, Pontryagin #'s are invariants of X up to oriented cobordism (say $X \sim_{\substack{\text{oriented}\\ \text{cob}}} X'$ if $\exists$ cpct oriented $W^{\dim X+1}$ with $\partial W \cong X \amalg \overline{X'}$ (orientation reversed) as oriented manifolds).

$\Leftrightarrow$ if $X = \partial W$ as oriented manifolds then all Pontryagin #'s are 0 (i.e., $X \sim_{\substack{\text{oriented}\\ \text{cob}}} \phi$).

Cor: $\underbrace{\mathbb{CP}^{2m}}_{\text{real } \dim_{\mathbb{R}} = 4m}$ is not the oriented boundary of any cpct oriented $(4m+1)$-dim'l m'fold.

(note in contrast that $\mathbb{CP}^1 = S^2 = \partial B^3$).

Also similar cor for $\coprod_r \mathbb{CP}^{2m}$, w/ same orientation for each copy.

(of course $\mathbb{CP}^{2m} \amalg \overline{\mathbb{CP}^{2m}}$ is $\partial(\mathbb{CP}^{2m} \times [0,1])$).

 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G_k(\mathbb{C}^\infty) \qquad G_k(\mathbb{R}^\infty)$

<u>Next</u>: want to compute the cohomology of $BU(k)$ resp. $BO(k)$. (why? any char. class of cplx. resp. real vector bundles of rank k is pulled back from a coh. class in $BU(k)$ resp. $BO(k)$ via classifying map, hence the computation would tell us what all possible such char. classes could be).

We'll focus on $BU(k) \overset{\shortparallel}{=} G_k(\mathbb{C}^\infty)$ ($BO(k)$ case, as usual is parallel provided we work w/ $\mathbb{Z}_2$ instead of $\mathbb{Z}$-coeffs.)

To analyze space, start w/

$$\begin{array}{c} E_{taut} \\ \downarrow \\ G_k(\mathbb{C}^\infty) \end{array}$$

The idea will be to use some form of splitting principle (a particular splitting map $s: Z \to BU(k)$) to embed $H^\bullet(G_k(\mathbb{C}^\infty))$ into $H^\bullet$(simpler space which can be computed)

The usual proof of the splitting principle produces a space $Z = \mathbb{F}(E_{taut})$ [fiberwise flags in E_{taut}] $= \mathbb{F}_k(\mathbb{C}^\infty)$. One option would be to use this space & compute $H^\bullet(\mathbb{F}(E_{taut}))$ explicitly by making use of Leray-Hirsch applied to various fibrations

e.g., $\mathbb{F}_k(\mathbb{C}^\infty) \longrightarrow \mathbb{CP}^\infty$ w/ fiber $\mathbb{F}_{k-1}$ --- see Hatcher's Alg. Topology book § 4.
$(L_1, \dots, L_k) \longmapsto L_1$

We'll take a shortcut by appealing to a different (somewhat simpler) splitting map (c.f. [Husemoller, <u>Fibre Bundles</u>]).

<u>Consider</u>: $X = \underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{k \text{ times}}$ On X we have the rank k vector bundle $E = L_{taut} \times \cdots \times L_{taut}$.

Equivalently, $E := \bigoplus_{i=1}^k \pi_i^* L_{taut}$, $\pi_i : X \to \mathbb{CP}^\infty$ proj. to i^{th} factor.

Since $BU(k)$ classifies rank k vector bundles, $\exists!$ (up to homotopy)

$$f_k : X \longrightarrow BU(k) \quad \text{with} \quad \underbrace{f_k^* E_{taut} = E := \bigoplus_{i=1}^k \pi_i^* L_{taut}}.$$

<u>Prop</u>: f_k is a splitting map for E_{taut}, i.e., ($f_k^* E_{taut}$ splits into line bundles) and f_k^* is injective.

<u>Pf</u>: Let $s: Z \to BU(k)$ be any splitting map for E_{taut} ($\exists$ by splitting principle), i.e.,

$s^* E_{taut} = L_1 \oplus \cdots \oplus L_k$ for $L_i \to Z$ and s^* is injective.

Since each L_i is a complex line bundle, it is classified by a map $g_i : Z \longrightarrow \mathbb{CP}^\infty$ (so $g_i^* L_{taut} = L_i$). Now consider $g = (g_1, \dots, g_k) : Z \longrightarrow (\mathbb{CP}^\infty)^k$, and let's observe that $g^*(E = \bigoplus_{i=1}^k \pi_i^* L_{taut}) = \bigoplus_{i=1}^k g^* \pi_i^* L_{taut} = \bigoplus_{i=1}^k g_i^* L_{taut} = \bigoplus_{i=1}^k L_i = s^* E_{taut}$.

In particular, $f_k \circ g : Z \to (\mathbb{CP}^\infty)^k \longrightarrow BU(k)$ classifies $s^*(E_{taut})$, because

$$(f_k \circ g)^*(E_{taut}) = g^* f_k^* E_{taut} = g^* E = s^* E_{taut}.$$

But $s: Z \to BU(k)$ classifies $s^* E_{taut}$ by definition. Since classifying maps are unique up to homotopy,

$$\Rightarrow f_k \circ g \simeq s.$$

$$\Rightarrow s^* = g^* f_k^*. \quad \text{But } s^* \text{ is injective.} \Rightarrow f_k^* \text{ is injective as desired.} \boxtimes$$

Using this, we have

Thm: Let $c_i := c_i(\overset{\text{rank } k}{E_{taut}}) \in H^{2i}(BU(k);\mathbb{Z})$. Then, the classes c_i are algebraically independent for $i = 1, \dots, k$, & moreover $H^\bullet(BU(k);\mathbb{Z}) \underset{\text{as rings}}{\cong} \mathbb{Z}[c_1, \dots, c_k]$ $(|c_j| = 2j)$.

Pf: Consider the map $f_k : (\mathbb{CP}^\infty)^k \to BU(k)$ which classifies $E = \bigoplus_{i=1}^k \pi_i^* L_{taut}$. By prev. Prop, $f_k^* : H^\bullet(BU(k);\mathbb{Z}) \to H^\bullet((\mathbb{CP}^\infty)^k;\mathbb{Z}) \underset{\text{Kunneth}}{\cong} \mathbb{Z}[h_1, \dots, h_k]$ is injective, $(|h_i| = 2 \text{ for each } i.)$

so need to calculate $\mathrm{im}(f_k^*)$. Now consider the action of the symmetric group Σ_k on $(\mathbb{CP}^\infty)^k$ by permuting factors. The induced action on $H^\bullet((\mathbb{CP}^\infty)^k)$ permutes $(h_1, \dots, h_n)$. Observe that E is invariant under such an action, that is, $\sigma^* E \cong E$ for any $\sigma \in \Sigma_k$. In particular, $f_k \circ \sigma$ still classifies E, so (by uniqueness of classifying maps up to homotopy) $f_k \circ \sigma \simeq f_k$ i.e., $\sigma^* f_k^* = f_k^*$. Hence the image of f_k^* lands in <u>symmetric polynomials</u> in $h_1, \dots, h_k$.

permutes h_i's.

Let's calculate $f_k^*(\underset{\overset{\|}{1+c_1+c_2+\dots+c_k}}{c(E_{taut})}) = c(f_k^*(E_{taut})) = c\left(\bigoplus_{i=1}^k \pi_i^* L_{taut}\right)$

$$\underset{\text{Whitney sum}}{=} \prod_{i=1}^k c(\pi_i^* L_{taut}) = \prod_{i=1}^k \pi_i^*(c(L_{taut})) = \prod_{i=1}^k \pi_i^*(1+h)$$

$$\overset{h_i := \pi_i^* h}{=} \underbrace{(1+h_1) \cdots\cdots (1+h_k)}.$$

Hence $f_k^* c_i = \deg 2i$ part of $\nearrow$ $= \left(\sum_{\substack{J \subseteq \{1, \dots, k\} \\ |J| = i}} \prod_{j \in J} h_j\right) = \sigma_i$ i^{th} elementary symmetric polynomial in $h_1, \dots, h_k$.

Fact: There are no alg. relations between any elementary symmetric polynomials, and any symmetric polynomial in $h_1, \dots, h_k$

can be uniquely written as a polynomial in $\sigma_1, \ldots, \sigma_k$

Using this, we learn that all symmetric polynomials in $\mathbb{Z}[h_1, \ldots, h_k]$ $\supseteq \operatorname{im}(f_k^*) \supseteq \{$subring of $\mathbb{Z}[h_1,\ldots,h_k]$ gen. by $\sigma_1, \ldots, \sigma_k\}$

$$\Rightarrow \operatorname{im}(f_k^*) \cong \mathbb{Z}[\sigma_1, \ldots, \sigma_k].$$

$\deg \sigma_i = 2i$.

with $c_i \overset{f_k^*}{\longmapsto} \sigma_i$.

Hence $H^*(BU(k); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_k]$. ∎

Cor: Each char. class $\phi : \mathrm{Vect}^k_{\mathbb{C}}(-) \longrightarrow H^*(-;\mathbb{Z})$ [of complex rank k bundles] must have the form $E \longmapsto q(c_1(E), \ldots, c_k(E))$ where q is a polynomial uniquely determined by the class. (q is the element of $\mathbb{Z}[c_1, \ldots, c_k] \cong H^*(BU(k); \mathbb{Z})$ given by taking $\phi(E_{taut})$).

Betti #'s : rank $H^i = b_i$.

Cor: $b_{2k+1}(BU(n)) = 0$, & $b_{2k}(BU(n)) = \operatorname{rk} H^{2k}(BU(2k))$

$$= \dim \left(\text{deg. } 2k \text{ part of } \mathbb{Z}[c_1, \ldots, c_n] \right)$$

$|c_i| = 2i$.

$= \#$ of monomials $c_1^{r_1} \cdots c_n^{r_n}$ of degree $2k = 2(r_1 + 2r_2 + 3r_3 + \cdots + n r_n)$. $\quad r_i \geq 0$

$= \#$ of n-tuples $(r_1, \ldots, r_n)$ w/ $k = r_1 + 2r_2 + \cdots + n r_n$.

$= \#$ of unordered partitions of k into at most n integers $\{k_1, \ldots, k_n\}$ ($k_1 \leq k_2 \leq \cdots \leq k_n$ & $\sum k_i = k$)

via $(r_1, \ldots, r_n) \longleftrightarrow \underbrace{r_n}_{k_1} \leq \underbrace{r_n + r_{n-1}}_{k_2} \leq \underbrace{r_n + r_{n-1} + r_{n-2}}_{k_3} \leq \cdots \leq \underbrace{r_n + \cdots + r_1}_{k_n}$.

Pf of Thm: Let $f_k : \underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{k} \longrightarrow BU(k)$ be the splitting map from above. (so $f_k^* E_{taut} \cong \bigoplus_{i=1}^k \pi_i^* L_{taut}$ / f_k^* injective.

By above:

$$f_k^* : H^*(BU(k);\mathbb{Z}) \xhookrightarrow{\text{injective}} H^*((\mathbb{CP}^\infty)^k;\mathbb{Z}) \underset{\text{künneth}}{\cong} \mathbb{Z}[h_1, \dots, h_k]$$

$|h_i| = 2$ for each i.

So just need to calc. $\operatorname{im}(f_k^*)$.

Now consider action of symmetric group $\Sigma_k \circlearrowright (\mathbb{CP}^\infty)^k$ permuting factors..

$\Rightarrow$ action on $H^*((\mathbb{CP}^\infty)^k)$ permutes $(h_1, \dots, h_k)$.

Note $E = \bigoplus_{i=1}^k \pi_i^* L_{taut}$ is invariant under such an action,

that is $\sigma^* E \cong E$ for any $\sigma \in \Sigma_k$.

$\Rightarrow f_k \circ \sigma$ still classifies E $\quad ((f_k \circ \sigma)^* E_{taut} \cong E)$

$\underset{\text{classifying map uniqueness}}{\Rightarrow}$ $f_k \circ \sigma \simeq f_k$ i.e., $\sigma^* f_k^* = f_k^*$

i.e., $\operatorname{im}(f_k^*)$ lands in <u>symmetric polynomials</u> in <u>$h_1, \dots, h_k$</u>.

Let's calculate $f_k^*(c(E_{taut})) = c(f_k^* E_{taut} = E)$

$$= c\left(\bigoplus_{i=1}^k \pi_i^* L_{taut}\right)$$

$$\underset{\substack{\text{Whitney}\\ \text{sum}}}{=} \prod_{i=1}^{k} c(\pi_i^* L_{taut}) = \prod \pi_i^* c(L_{taut})$$

$$= \prod \pi_i^* (1+h)$$

$$= \prod (1+h_i)$$

$$= \underbrace{(1+h_1)\cdots(1+h_k)}$$

so $f_k^* c_i$ = deg $2i$ part of ↑

$$= \left(\sum_{\substack{J \subseteq \{1,\dots,k\} \\ |J| = i}} \prod_{j \in J} h_j \right) = \sigma_i =$$

i^{th} elementary sym. poly. in $h_1, \dots, h_k$

Fact: There are no algebraic relations between elementary sym. polys, & every symmetric polynomial can be uniquely written as a poly. in $\sigma_1, \dots, \sigma_k$

$\Rightarrow$

$$=$$

$\{$all symmetric polys.$\}$ $\supseteq \mathrm{im}(f_k^*) \supseteq \{$subring of $\mathbb{Z}[h_1, \dots, h_k]$ gen. by $\sigma_1, \dots, \sigma_k\}$

$= \qquad = $

$$\cong \mathbb{Z}[\sigma_1, \dots, \sigma_k]$$

$\deg \sigma_i = 2i$.

w/ $c_i \xmapsto{f_k^*} \sigma_i$. As f_k^* injective,

$$\Rightarrow H^\bullet(BU(k); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_k]$$