Last time: computed $H^*(BU(k);\mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_k]$, $|c_i| = 2i$.

The same arguments apply to compute $H^*(BO(k);\mathbb{Z}/2)$ (using $\mathbb{RP}^\infty$ instead of $\mathbb{CP}^\infty$ etc. as usual)

$\Rightarrow$ <u>Thm</u>: $H^*(BO(k);\mathbb{Z}/2) \cong \mathbb{Z}[w_1, \dots, w_k]$ where $w_i := w_i(E_{taut})$, $|w_i| = i$.

(in particular w_i are all alg. independent).

$\Rightarrow$ all char. classes of real vect. bundles of rank k taking values in $H^*(-;\mathbb{Z}/2)$ are polynomials in the Stiefel-Whitney classes.

We won't spell out the details, but a more involved computation shows that, modulo certain 2-torsion elements

$$H^*(BO(k);\mathbb{Z}) \cong \mathbb{Z}[p_1, \dots, p_{\lfloor \frac{k}{2} \rfloor}] \ (\text{mod } 2\text{-torsion})$$

Pontryagin classes of E_{taut}.

(beginning of another possible paper topic!)

(2-torsion given by certain polynomials in Stiefel-Whitney classes).

We can also look for char. classes of vector bundles equipped with more structure, e.g., an <u>orientation</u>.

This is what we'll now do. <u>Goal</u>: to define the <u>Euler class</u> of oriented $E \to X$ using a natural class on an oriented bundle called its <u>Thom class</u>.

lives in $H^*(X;\mathbb{Z})$

lives in $H^*(E, E \setminus \underline{0})$.

Recall that an <u>orientation</u> of a vector space V dimension n is

$V^0 := V \setminus 0$

an equivalence class of basis $\mathcal{B} = (v_1, \dots, v_n)$ modulo $\mathcal{B} \sim \mathcal{B}'$ if $\mathcal{B} = T\mathcal{B}'$ w/ $\det(T) > 0$ OR a generator μ_V of $H_n(V, \overline{V^0}; \mathbb{Z})$

(<u>Exercise</u>: why is this true? Assign to a basis $\mathcal{B} = (v_1, \dots, v_n)$ a linear n-simplex $[e_0, \dots, e_n] = \sigma_{\mathcal{B}}$ in V w/ barycenter in 0 w/ $\overrightarrow{e_0 e_1} = \vec{v}_1$, $\overrightarrow{e_1 e_2} = \vec{v}_2$, ___ .

$\Rightarrow$ a gen. for $H_n(V, V^0; \mathbb{Z})$

check now that if $\mathcal{B} \sim \mathcal{B}'$ then $[\sigma_{\mathcal{B}}] = [\sigma_{\mathcal{B}'}]$.

if $\mathcal{B} \not\sim \mathcal{B}'$ then $[\sigma_{\mathcal{B}}] = -[\sigma_{\mathcal{B}'}]$.
)

Similarly the cohomology group $H^n(V, V_0; \mathbb{Z})$ has a preferred generator u_V if V is oriented, by require $\langle u_V, \mu_V \rangle = +1$.

rank

i.e., can find $\{U_\alpha\}$ a trivializing cover s.t. clutching fns.

We say a vector bundle $E^n \to X$ is orientable if

file values in $GL(n)$ OR $\mathrm{Frame}(E)$ has a reduction to $GL(n)^+$

E admits a reduction of structure group to $GL(n)^+ \subset GL(n)$

$\iff$ $\exists$ a section of $\mathrm{Frame}(E) \times_{GL(n)} \left(GL(n)/GL^+(n) \cong \mathbb{Z}/2 \right)$.

$\iff$ $\exists$ a section of the bundle whose fibers are $\{H^n(E_x, (E_x)_0; \mathbb{Z})\}_{x \in X}$, generating each fiber.

construct this bundle? analogous to the bundle $M_R \to M$ we constructed earlier, or: if $E = \mathrm{Frame}(E) \times_{GL(n)} \mathbb{R}^n$, then consider $\mathrm{Frame}(E) \times_{GL(n)} \underbrace{H^n(\mathbb{R}^n, (\mathbb{R}^n)_0; \mathbb{Z})}_{GL(n) \text{ acts induced by action on } \mathbb{R}^n.}$

$\iff$ A choice $\{u_x \in H^n(E_x, (E_x)_0; \mathbb{Z})\}_{x \in X}$ varying 'continuously';

meaning for each x $\exists$ $U \subseteq X$ open containing x & $u_U \in H^n(E|_U, (E|_U)_0; \mathbb{Z})$

$E|_U - \underline{0}|_U.$

restricting along $(E_y, (E_y)_0) \xrightarrow{i_y} (E|_U, (E|_U)_0)$ to u_y for each $y \in U$..

An <u>orientation</u> on E is any such choice of section/restriction of structure group as above.

image of 0-section

$E^0 := E \setminus \underline{0}$

rank n

<u>Def</u>: A <u>Thom class</u> for an oriented vector bundle $E \to X$ is a class $u \in H^n(E, E^0)$ implicitly $\mathbb{Z}$-coeffs.

with $i_x^* u = u_x$ for each $x \in X$ where $i_x : E_x \hookrightarrow E$ incl. of a fiber

(can also ask for a Thom class w/ $\mathbb{Z}/2$-coeffs, but then don't require E to be orientable; following results all hold w/ $\mathbb{Z}/2$ coeffs. for bundles which are not nec. orientable!

<u>Lem</u>: If such a u exists, then

(a) (<u>Thom isomorphism theorem</u>) The map $\Psi : H^\bullet(X) \xrightarrow{\cong} H^{\bullet+n}(E, E^0)$ is an iso.

$\alpha \longmapsto u \smile \pi^* \alpha$

$\pi^*\alpha \in H^\bullet(E)$, then use rel cup product.

i.e., $H^i(X) \xrightarrow{\cong} H^{i + \mathrm{rank}(E)}(E, E^0)$ i.e.,

- $H^k(E, E^0) = 0$ for $k < \mathrm{rank}(E)$.
- Any element of $H^n(E, E^0)$ has the form

  $\pi^* f \smile u = f \cdot u$ for f a function on X $f : X \to \mathbb{Z} \leftrightarrow C^0(X; \mathbb{Z})$ which is <u>locally constant</u> $\longleftrightarrow$ $\delta f = 0$.

(b) In particular, by , such a u is <u>unique</u>.. (immediate cor. of (a)).

(any $\tilde{u} \in H^n(E, E^0)$ is of the form $\tilde{u} = f \cdot u$, but now

$$\begin{array}{ccc} i_x^* \tilde{u} & = & u_x \\ \shortparallel & & \| \\ i_x^*(f \cdot u) & & \\ \shortparallel & & \\ f(x) i_x^* u & = & f(x) u_x \end{array} \Rightarrow f(x) = 1 \ \forall x. \quad)$$

Pf of Lemma:

Observe that one can extend Leray-Hirsch theorem to study fibration pairs over B, i.e., pairs of fibrations $\begin{array}{c}(P, P') \\ \downarrow \\ B\end{array}$ $P' \subset P$ whose fibers are (F, F'). $F' \subset F$ Leray-Hirsch in such a setting says:

If $H^*(F, F')$ is free + fin. gen. in each degree and $H^\bullet(P, P') \xrightarrow{res^*} H^\bullet(F, F')$ is surjective, then choosing classes $\{c_j \in H^{n_j}(P, P')\}$ restrict to a given generating basis $\{\gamma_j \in H^{n_j}(F, F')\}$

"cohomology extension of fiber"

determines an iso. of $H^\bullet(B)$-modules
$$H^*(B) \otimes_R H^\bullet(F, F') \xrightarrow{\cong} H^\bullet(P, P')$$
$$b \otimes \gamma_j \longmapsto \pi^* b \cup c_j.$$

"$b \cdot c_j$ using module str. of $H^\bullet(B)$ on $H^\bullet(P, P')$".

(Pf is same, or can be deduced from absolute case by studying LES of a pair. — exercise).

Our case: $\begin{array}{c}(P, P') \\ \downarrow \\ B\end{array} = \begin{array}{c}(E, E^0) \\ \downarrow \\ X\end{array}$. Note that $H^*(F, F') = H^*(E_x, E_x^0) \cong H^*(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$
$$= \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else.} \end{cases}$$ free, fin. gen. in each degree.

Let γ_1 be the basis u_x coming from orientation on E.

By hypothesis, $\exists$ 'Thom class', i.e., a class $c_1 = u$ w/ $c_1|_{(F,F')} = \gamma_1$ so restr. map is surjective.

not Chern class! just c_j w/ $j=1$ as above.

Using this choice of coh extension of fiber, rel. L-H $\Rightarrow$

$$\begin{array}{ccc} b \otimes \gamma_1 & \longmapsto & \pi^* b \cup c_1 = \boxed{\pi^* b \cup u.} \\ H^\bullet(X) \otimes_R \underbrace{H^\bullet(E_x, (E_x)^0)}_{\mathbb{Z} \text{ in degree rank}(E)} & \xrightarrow{\cong} & H^\bullet(E, E^0) \\ \uparrow & & \\ \| \wr & & \\ H^{\bullet - \text{rank}(E)}(X) & & \\ b & & \end{array}$$

This establishes Thom iso. theorem.

Existence?

Thm: If $E \to X$ is orientable, a Thom class always exists. (by above $\exists!$ Thom class for each choice of orientation).

Pf sketch: Inductive argument. (only outline given in class).

Step 1: A Thom class always exists over $E|_U$, $U \subset X$ if $E|_U$ is trivial.

In that case: $H^*(E|_U, (E|_U)^\circ) \cong H^*(U \times \mathbb{R}^n, U \times (\mathbb{R}^n \setminus 0)) \underset{\text{Künneth}}{\cong} H^*(U) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$

$$u \longleftarrow 1 \otimes u_{\mathbb{R}^n}$$

choice of orientation of $\mathbb{R}^n$.

(exercise: check u is indeed a Thom class for orientation induced by $u_{\mathbb{R}^n}$).

Step 2: Say $E|_{U \cup V}$ orientable, $\exists$ Thom classes u_U for $E|_U$ and u_V for $E|_V$ compatible w/ chosen orientation of E, i.e., $(u_U)_x = (u_V)_x$ when $x \in U \cap V$.

Then $\exists$ Thom class $u_{U \cup V}$ for $U \cup V$ which restricts to u_U and u_V.

by M-V exact sequence for (E, E°) restr. to $U, V, U \cap V$:

$$H^{n-1}(\cancel{E|_{U\cap V}, E^\circ|_{U\cap V}}) \longrightarrow \underset{(*)}{H^n(E|_{U\cup V}, E^\circ|_{U\cup V})} \longrightarrow \underset{(u_U\ ,\ u_V)}{H^n(E|_U, E^\circ|_U) \oplus H^n(E|_V, E^\circ|_V)}$$

0 b/c $\exists$ Thom class over $U \cap V$ & Thom iso. applies, $n-1 < \mathrm{rank}(E) = n$

$$\longrightarrow H^n(E|_{U\cap V}, E^\circ|_{U\cap V}) \longrightarrow H^{n+1}(\cdot\,\cdot) \ldots$$

$(u_U)|_{U\cap V} - (u_V)|_{U \cap V} = 0$ by hypothesis.

By exactness, $\exists$ $u_{U \cup V}$ in $(*)$ restricting to u_U and u_V as desired.

Step 3: Inductively as in other proofs use steps 1+2 to deduce existence of Thom classes when X is a finite dim'l CW complex.

(by decomposing $X^k = X^{k-1} \cup \amalg \{e^k_\alpha\}$ & applying to $U = \mathring{X}^k = X^{k-1} \cup \amalg \{\mathring{e}^k_\alpha\}$ ($\simeq X^{k-1}$),

$V = \amalg \{\mathrm{int}(e^k_\alpha)\}$, etc.)

Step 4: extend to all CW complexes by 'finite dim'l approx' of any given class.

Step 5: extend to X any space (by 'CW approximation'). $\boxtimes$.

Euler class:

Given E rank n, oriented, real vector bundle, have an inclusion $(X, \phi) \xhookrightarrow[i_0]{(0, \phi)} (E, E^\circ)$. (zero section)

X

Def'n: For $E \to X$ as above with $u \in H^n(E, E^0)$ its Thom class, the Euler class of E is:

$$e(E) := i_X^* u \in H^{n}(X; \mathbb{Z})$$

rank(E)

We can think of $e(E)$ as the image of Thom class under

$$\begin{tikzcd} X \arrow[r, "0", "\text{homotopy equiv.}"'] \arrow[rr, bend right, "i_X"'] & E \arrow[r, "\text{incl.}"] & (E, E^0), \end{tikzcd}$$

$(E \leftrightarrow (E, \phi))$.

i.e., $H^n(E, E^0) \xrightarrow{res^*} H^n(E) \xrightarrow{\cong} H^n(X)$,
$u \longmapsto e(E)$.

Properties of the Euler class

Lemma: If $E \to X$ has a nowhere vanishing section $s: X \to E$ (using a metric on E: $E \cong \underline{\mathbb{R}} \oplus E'$, $(\underline{\mathbb{R}})^\perp$)
then $e(E) = 0$.

"$e(E)$ obstructs existence of a non-vanishing section"

(i.e., $e(F \oplus \underline{\mathbb{R}}) = 0$; note in contrast $w_i/p_i(F \oplus \underline{\mathbb{R}}) = w_i/p_i(F)$).

Pf: Note: Any two sections $s, s' \in \Gamma(E)$ are homotopic as maps $X \to E$ via homotopy $(1-t)s + ts'$.

In particular, if $E \to X$ has a non-vanishing section s, then $s \simeq i_X = (\underline{0}, \phi)$ as maps

$$(X, \phi) \to (E, E^0)$$

$\Rightarrow e(E) = i_X^* u = s^* u$, but since s is nowhere vanishing, s factors as

$$\begin{tikzcd} (X, \phi) \arrow[rr, "s := (s, \phi)"] \arrow[dr, "s\ (b/c\ s_x \neq 0\ \forall x)."'] & & (E, E^0) \\ & (E^0, E^0) \arrow[ur, "\text{incl.}"'] & \end{tikzcd}$$

i.e., s^* factors through $H^*(E_0, E_0) = 0$, so $e(E) = 0$. $\boxtimes$.

Say $\underset{\substack{\text{rank } n\\ = n_1+n_2}}{E} = \underset{\text{rank } n_1}{E_1} \oplus \underset{\text{rank } n_2}{E_2}$ with each E_i oriented $\Rightarrow$ induces a canonical orientation of E.

(fiberwise: if $(e_1, \dots, e_{n_1})$ oriented basis of $(E_1)_x$ & $(f_1, \dots, f_{n_2})$ oriented basis of $(E_2)_x$

$\Rightarrow$ declare $(e_1, \dots, e_{n_1}, f_1, \dots, f_{n_2})$ to be an oriented basis of E_x

$\Rightarrow$ a map $or((E_1)_x) \times or((E_2)_x) \to or((E)_x)$).

Using these compatible orientations to define Euler classes:

Prop: $e(E) = e(E_1) \cup e(E_2)$.

Rmk: Similar to, but different in practice from Whitney sum formula for total Chern/Stiefel-Whitney/Pontryagin classes.

note: whereas $w/p(\underline{\mathbb{R}}) = 1$ (resp. $c(\underline{\mathbb{C}}) = 1$), $e(\underline{\mathbb{R}}) = 0$, i.e., is not a unit.

So this formula can't always be used e.g. to solve for $e(E_1)$ given $e(E_2)$ & $e(E = E_1 \oplus E_2)$.

Pf of proposition:

Let $\pi_i : E \to E_i$ (fiberwise) projection onto ith factor, $i = 1, 2$.

gives: $\overline{\pi}_1 : (E, E \setminus E_2) \to (E_1, \underset{E_1 \setminus 0}{\underset{\shortparallel}{E_1^0}})$ $\quad \overline{\pi}_2 : (E, E \setminus E_1) \to (E_2, E_2^0)$.

Let $u_i \in H^{n_i}(E_i, E_i^0)$ be the Thom classes of E_i $i = 1, 2$.

Lemma: The Thom class for E (using given orientation), u, satisfies:

$$u = \overline{\pi}_1^* u_1 \cup \overline{\pi}_2^* u_2 .$$

rel. cup prod. $H^{n_1}(E, E \setminus E_2) \times H^{n_2}(E, E \setminus E_1) \to H^{n = n_1 + n_2}(E, \underset{(E \setminus E_1) \cup (E \setminus E_2)}{\overset{\shortparallel}{E^0}})$.

By uniqueness of Thom classes, it suffices to verify both sides are Thom classes & agree on any given fiber $E_x = (E_1)_x \oplus (E_2)_x$.

(Exercise: check that the induced orientation on a direct sum $E_x = (E_1)_x \oplus (E_2)_x$, thought of as an elt. of $H^n(E_x, E_x^0)$, is induced from the ones $(u_i)_x \in H^{n_i}((E_i)_x, (E_i)_x^0)$ $i = 1, 2$ precisely by $\pi_1^*(u_1)_x \cup \pi_2^*(u_2)_x$.)

Using lemma: $e(E) := i_x^* u$ where

$$\begin{tikzcd}
 & (E, E \setminus E_1) \arrow[d, "\text{ind.}"] \\
(X, \phi) \arrow[ur, "i_x^1"] \arrow[r, "i_x"] \arrow[dr, "i_x^2"'] & (E, E^0) \\
 & (E, E \setminus E_2) \arrow[u, "\text{ind.}"']
\end{tikzcd}$$

$$\Rightarrow i_x^* u \overset{\text{(lem)}}{=} i_x^* (\overline{\pi}_1^* u_1 \cup \overline{\pi}_2^* u_2)$$

$$= (i_x^{1*} \overline{\pi}_1^* u_1) \cup (i_x^{2*} \overline{\pi}_2^* u_2)$$

(exercise)

$= e(E_1) \smile e(E_2)$.

$\Rightarrow$ it must be pulled back from $H^n(BSO(n);\mathbb{Z})$, where $BSO(n) = BGL^+(n) =$ "classifying space of rank n oriented bundles"

The Euler class is a (natural) invariant of (E, ω), though we sometimes leave ω implicit; & note

bundle orientation

$$e(E,-\omega) = -e(E,\omega).$$

In particular, since $(-id): \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ is orientation reversing when n is odd.

$\Rightarrow$ $(E,\omega) \underset{\text{as oriented bundles}}{\overset{(-id)}{\cong}} (E,-\omega)$ when $\mathrm{rank}(E) = \text{odd}$.

$\Rightarrow$ $e(E,\omega) = -e(E,\omega)$ when $\mathrm{rank}(E)$ is odd.

Cor: If $\mathrm{rank}(E)$ is odd, then $2e(E,\omega) = 0$ (i.e., $e(E,\omega)$ is 2-torsion).

(will be forced to be zero if no 2-torsion in that coh. group).

Cor of Euler class: Say (E,ω) even-dim'l oriented bundle & $2e(E,\omega) \neq 0$. Then, E cannot split as sum of two odd rank oriented bundles.

note: on oriented m'fold, $2e(TM) \neq 0 \Rightarrow e(TM) \neq 0$.

If M oriented manifold, we'll call $e(M) := e(TM)$ Euler class of M. $\in H^{\dim(M)}(M;\mathbb{Z})$

(using Cor. above)

Exercise: M oriented manifold with $e(M) \neq 0$. Then, TM doesn't admit an odd-dim'l subbundle $\xi \subset TM$. (In particular, $\dim(M)$ is even)

(Hint: case (i): show $\nexists$ orientable odd rank $\xi \subset TM$

(ii) Say $\exists$ $\xi \subset TM$ odd, non-orientable; pull back to a 2-fold cover of M over which ξ orientable to reduce to (i)).

We can also take the characteristic # associated to $e(M)$ (say M cpct, oriented), and:

Thm: M cpct, oriented. $\langle e(M), [M] \rangle = \chi(M)$ $\leftarrow$ Euler characteristic of M, which can e.g., be defined as $\chi(M) = \sum (-1)^i \dim H^i(M)$.