M. Abouzaid: On generating Fukaya Categories.

Motivation:

Conj: (Kontsevich '08): X Stein manifold, Λ isotropic skeleton.

$\exists$ $\mathcal{A}$ cosheaf of A_∞ alg. on Λ

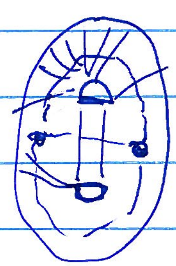
$\mathcal{F}^{\cdot}_{cpt}(X) \hookrightarrow \mathrm{mod}\text{-}\mathcal{A}$.

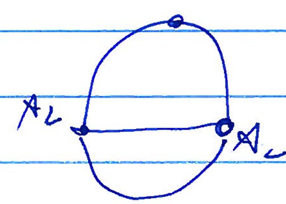
$X = T^*Q$ ← smooth closed manifold.

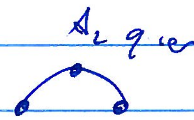
$\mathcal{A}$ constant cosheaf $\underline{\underline{\mathrm{Vect}}}_k$ ← graded

$\mathcal{F}_{cpt}(X) \hookrightarrow \mathcal{L}oc(Q)$.

Picture in dim. 1

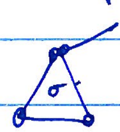






Setup: • X Liouville domain (exact), ∂X contact.

• Q simplicial complex.

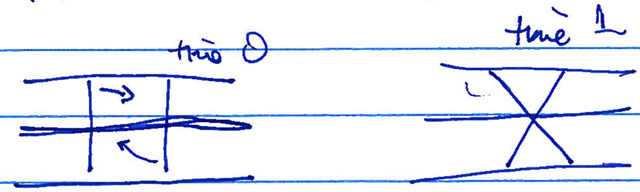


• smooth family L_σ of exact Lagrangians w/ Leg. ∂ for every simplex, and an inclusion $L_{\sigma/\tau} \subset L_\tau$ whenever $\tau \subset \partial\sigma$.

Key assumption: (local constancy, constructibility)

∂L_σ a family of Legendrians in ∂X, is intertwined by diffeos. in ∂X.

Explanation: Avoid σ = edge

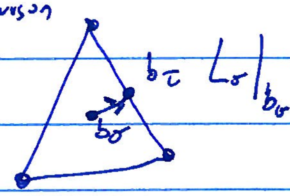


(Rules could maybe further subdivide to deal w/ this)

First result:

There exists a sheaf of A_∞ categories on a subdivision of Q whose value on a simplex σ is the Fukaya category $\mathcal{B}(\sigma)$ with object $L_\sigma|_{\text{barycenter}}$.

ignore subdivision first.



$L_\sigma|_{b_\sigma}$ HF^*(components) are the morphism spaces in $\mathcal{B}(\sigma)$.

Given an inclusion $\tau \subset \partial\sigma$, we have an A_∞ functor $\mathcal{B}(\sigma) \to \mathcal{B}(\tau)$.

(need constancy conditions to get continuation maps)

Given a triple $\tau \subset \partial\sigma$, $\rho \subset \partial\tau$, have

$$\begin{tikzcd} \mathcal{B}(\sigma) \arrow[r] \arrow[rr, bend right, "\text{commutes}"] & \mathcal{B}(\tau) \arrow[r] & \mathcal{B}(\rho) \end{tikzcd}$$

because we subdivided appropriately, otherwise, needed homotopy coherence.

e.g.

subdivision: this category becomes bigger (two objects).

Now, given $K \subset X$ compact Lagrangian,

consider $HF^*(\cancel{K}, L_\sigma|_{b_\sigma})$ module over $\mathcal{B}(\sigma)$.

ex: $$\begin{array}{c} HF^*(K, L_\sigma) \otimes HF^*(L_\sigma|_{b_\sigma}, L_\sigma|_{b_\sigma}) \\ \Big\downarrow \quad \text{composition maps} \\ HF^*(K, L_\sigma|_{b_\sigma}) \end{array}$$

$\rightsquigarrow \mathcal{L}(\sigma)$ notation for this module.

Second result:

$\mathcal{L}(\sigma)$ forms an ∞-cosheaf of modules over $\mathcal{B}(\sigma)$.

More precisely, there should be a map

$$\mathcal{L}(\sigma) \otimes_{\mathcal{B}(\sigma)} \mathcal{B}(\tau) \to \mathcal{L}(\tau).$$

i.e. exists a map

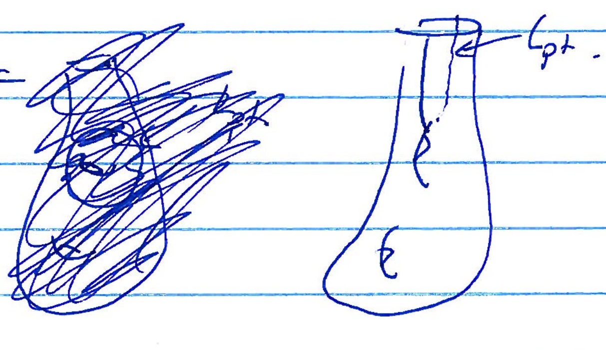
$$HF^*(K, L_\sigma) \otimes HF^*(L_\sigma, L_\tau) \to HF^*(K, L_\tau).$$

The analogous diagrams don't strictly commute, so need to put in all data of all homotopies.

~~How~~ Q: ~~What~~ How to compute $HF^*(K, K)$ starting from $\mathcal{L}$?

A: In general, can't.

ex. $Q = pt.$, $X =$

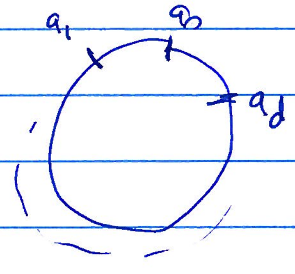


Review of HH_*:

Given $\mathcal{B}$, $\rightsquigarrow HH_*(\mathcal{B}) = H_*(CC_*(\mathcal{B}), b)$

generators

$CC_*(\mathcal{B}) = \bigoplus_{d \geq 1}^{\infty} \mathcal{B}^{\otimes d}$.

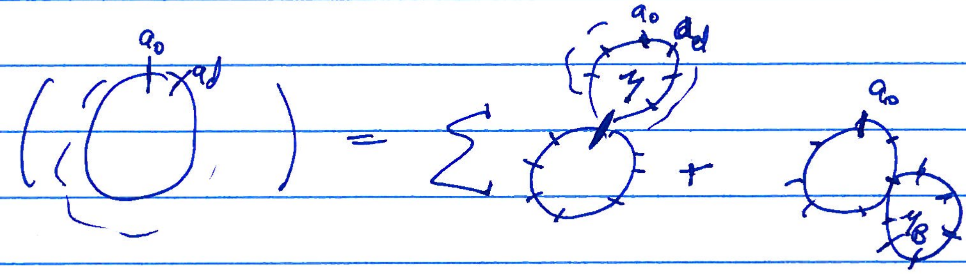


, $a_i \in \mathcal{B}$.

differential takes any interval within circle and collapses it using

$\mu_{\mathcal{B}}^k \leftarrow$ higher product ($k \geq 1$)

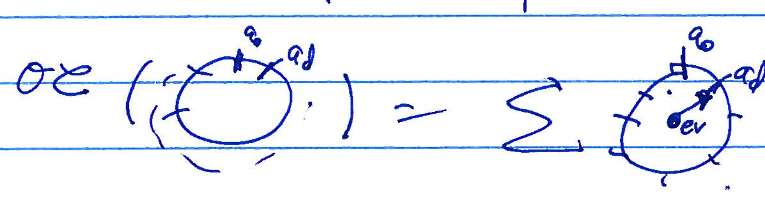
Picture :



- There exists a natural map

$$\mathcal{OC}: HH_*(\mathcal{B}_\sigma) \longrightarrow H_*(X, \partial X)$$

open-closed map



moduli space of hol. disks w/ boundary conditions $a_0, a_1, \dots, a_d$, and one interior marked pt.

$ev_*: \mathcal{M}_1(a_0, \dots, a_d) \longrightarrow \cancel{C_*(X, \partial X)}\ X$ (Rmk: could use $\mathbb{Z}$ coeffs, but do $\mathbb{Z}/2\mathbb{Z}$ now).

pass to chains, and evaluate

$ev_*[\mathcal{M}(a_0, \dots, a_d)]$

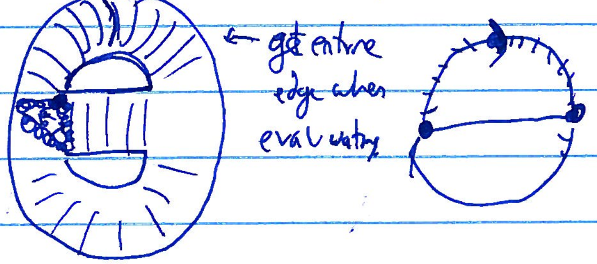
Third Result: these maps can be organized into a map

$$H_*(Q, HH_*(\mathcal{B})) \to H_*(X, \partial X)$$

H.OC $\quad H_*(Q, HH_*(\mathcal{B})) \to H_*(X, \partial X)$

$$HH_*(\mathcal{B}_\sigma) \to HH_*(\mathcal{B}_\tau)$$

vec. spaces, form a cosheaf



circle of morphisms on vertex

, so applying OC gives us entire triangle.

So, decompose the space as a union of

1. - fibers
2. - images of hol. disks w/ the right boundary conditions.

i.e. $[X] \in H_*(X, \partial X)$ lies in the image of H. OC.

This is the answer to the original question.

Fourth result! $HF^*(k,k)$ can be recovered from $\mathcal{L}$ if $[X]$ lies in the image of H. OC.

<u>Recall:</u> $\mathcal{L} \sim HF^*(K, L_\sigma)$

$\uparrow$ left module (strange perspective).

$\mathcal{R} \sim HF^*(L_\sigma, K)$.

Precise statement:

$$HF^*(K,K) \cong \mathrm{Tor}_{\mathcal{B}}(\mathcal{L}, \mathcal{R})$$

Everything is contained in a commutative diagram.

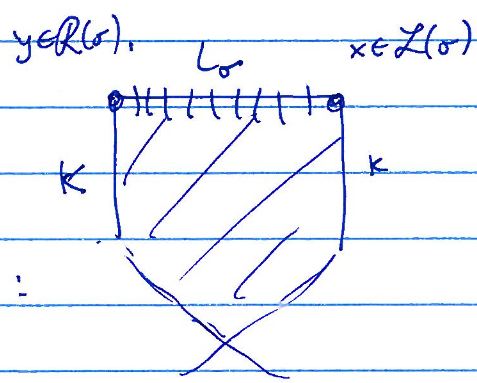
$HF^*(K,K) \otimes [Q]$ implies

$$\begin{tikzcd}
HF^*(K,K) \otimes H_*(Q, HH_*(\mathcal{B})) \arrow[r, "\text{injective}"] \arrow[d, "\mathbb{1} \otimes H_* \mathcal{OC}"] & H_*(Q, \mathrm{Tor}_{\mathcal{B}}(\mathcal{L}, \mathcal{R})) \arrow[d, "\text{surjective}"] \\
HF^*(K,K) \otimes H_*(X, \partial X) \arrow[r] \arrow[d, equal] & HF^*(K,K) \arrow[d, equal] \\
H^\cdot(K) \otimes H^\cdot(X) \arrow[r] \arrow[r, bend right, "\mathrm{id}."'] & H^\cdot(K)
\end{tikzcd}$$

id.

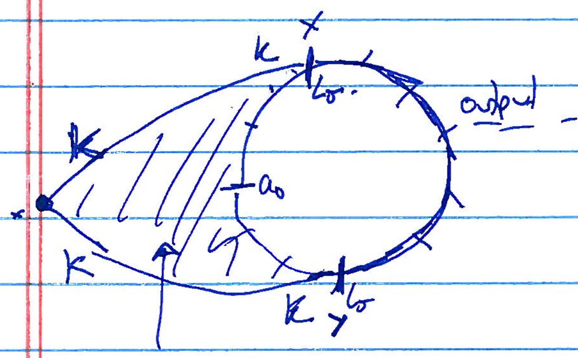
Look at right vertical map:

$$\mathrm{Tor}_{\mathcal{B}(\sigma)}(\mathcal{L}(\sigma), \mathcal{R}(\sigma)) \sim H_*\left(\bigoplus_i \mathcal{L}(\sigma) \otimes \mathcal{B}(\sigma)^{\otimes i} \otimes \mathcal{R}(\sigma), d\right)$$



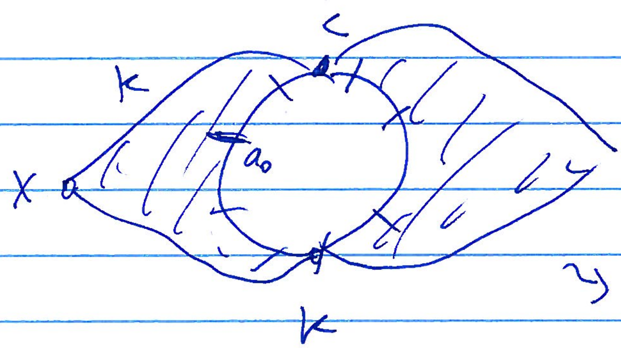
map is:

what's the top horizontal map?

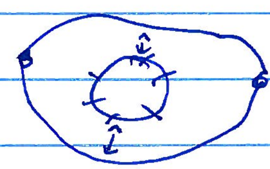


$\bar\partial u = 0$ (hol. disks

composition of top horizontal & right vertical!



~> smooth



now, shrink disc, get other part of commutative diagram