S. Guillermou -

Microlocal theory of sheaves & S.G.

Intro microsupport

$M$ manifold, $\mathcal{F}$ sheaf

$SS(\mathcal{F}) \subset T^*M$ closed and conic, describes singularity of $\mathcal{F}$.

Thm: SS(F) coisotropic

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & V \\ \cap & & \cap \\ \dot{T}^*M & & \dot{T}^*N \end{array}$$

$\varphi$ sympl iso, homogeneous, i.e. $\varphi(x, \lambda\xi) = \lambda \cdot \varphi(x, \xi)$

$(x_0, \xi_0) \in U$, $(y_0, \eta_0) = \varphi(x_0, \xi_0)$. (for $\mathcal{F}$ around $x_0$, we can build $\mathcal{F}'$ around $y_0$ s.t. $SS(\mathcal{F}') = \varphi(SS(\mathcal{F}))$ around $\xi_0, \eta_0$.

Recently: • Nadler-Zaslow

• Tamarkin.

Propose to work in reverse direction: Assume you have $\Lambda \subset \dot{T}^*M$ ($T^*M \setminus M$) remove 0 section Lagrangian submanifold, conic.

Find some canonical sheaf $\mathcal{F}$ on $M$ with microsupport $SS(\mathcal{F}) = \Lambda \cup M$.

$\rightarrow$ deduce properties of $\Lambda$.

Microsupport:

$M$ manifold, $k$ ring.

sheaf $\mathcal{F}$ on $M$ = sheaf of $k$-modules, i.e. $\forall U \subset M$ we have a $k$-module of sections $\Gamma(U, \mathcal{F}) = \mathcal{F}(U) = H^0(U, \mathcal{F})$.

- restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subset U$

+ gluing conditions.

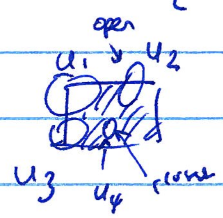
example: $Z \subset M$ closed subset

$k_Z$ $\quad k_Z(U) = \{ f : U \cap Z \longrightarrow k, \text{ locally constant} \}$

$\qquad\qquad \simeq k^{\pi_0(U \cap Z)}$

• $Z \subset M$ locally closed $\quad Z = \overset{\text{open}}{U} \cap \overset{\text{closed}}{W}$

$k_Z(U) = \{ f : U \cap Z \to k, \text{ loc. } \cancel{\text{closed}} \text{ const.}, \ \operatorname{supp}(f) \text{ closed in } U \}$.

$Z =$ 


$k_Z(U_i) = \begin{cases} 0 & i = 1, 2 \\ k & i = 3, 4 \end{cases}$

sheaves on $M$ = abelian category

for $U \subset M$, gives $F \longmapsto \Gamma(U, F)$ is left exact

- derived functors

$H^i(U, F)$

ex: $F = k_Z$ $\quad Z$ closed $\Rightarrow H^i(U; k_Z) = H^i(U \cap Z.)$

$Z$ locally closed

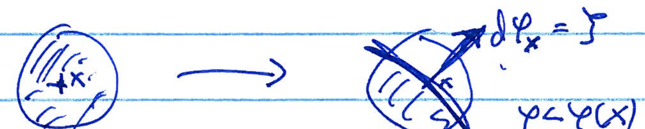
we can deduce $H^i(U; k_Z)$ by excision LES.

Def. (K.S. 8.2): $M$ manifold

$F$ sheaf on $M$ (or even $F \in D^b(k_M)$).

$(x_0, \xi_0) \in T^*M$, $\quad (x_0, \xi_0) \notin SS(F)$ if $\forall (x, \xi)$ in some nhood,

for any $\varphi : M \to \mathbb{R}$, $d\varphi_x = \xi$



$$\varinjlim_{U} H^i(U;\mathcal{F}) \xrightarrow{\sim} \varinjlim_{U \ni x} H^i(U \cap \{\varphi < \varphi(x)\};\mathcal{F}).$$

$$\| $$

$$\mathcal{F}_x.$$

~~in particular, $\mathcal{F}_x$ is zero in neighborhood of $x_0$).~~

Rmk: (1) If $(x_0, 0) \notin SS(\mathcal{F})$, then $\mathcal{F} = 0$ along $x_0$.

so: • $\pi_M(SS(\mathcal{F})) = \operatorname{supp} \mathcal{F}$

$\pi_M : T^*M \to M$.

• $SS(\mathcal{F})$ is closed and conic, and $\pi_M(SS(\mathcal{F})) = SS(\mathcal{F}) \cap M$.

Prop: "Morse Lemma": $\mathcal{F} \in D^b(k_M)$

• $\varphi : M \to \mathbb{R}$ s.t. $\varphi|_{\operatorname{supp}\mathcal{F}}$ proper.

$$H^i(\varphi^{-1}((-\infty, b)), \mathcal{F}) \xrightarrow[R^i_{b,a}]{} H^i(\varphi^{-1}((-\infty, a)), \mathcal{F}),$$

$a < b$

• if $\forall x \in \varphi^{-1}([a,b))$ $(x, d\varphi_x) \notin SS(\mathcal{F})$,

then $R^i_{b,a}$ is an isom. $\forall i$.

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Example:

• $N \subset M$ submanifold

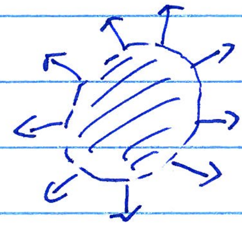
$SS(k_N) = T_N^*M$

• $U \subset M$ open subset, $\partial U$ smooth

exterior/
outer conormal bundle

Then $SS(k_U) = (T^{*,e}_{\partial U} M) \cup \bar{U}$

e.g.



locally $U = \{\varphi < 0\}$

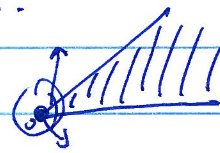
$T^{*,e}_{\partial U} M$ ~~$SS(k_U)$~~ $= \{(x, \lambda d\varphi_x);\ \lambda > 0\}$
$\qquad\qquad\qquad\qquad \varphi(x) = 0$

$SS(k_{\bar{U}}) = a(SS(k_U))$

$a(x, \xi) = (x, -\xi)$

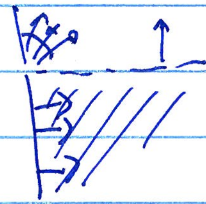
<u>example</u>: $Z$ convex cone in $M$ vector space, <u>closed</u>

e.g.

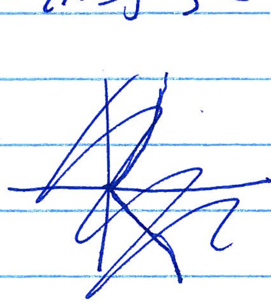


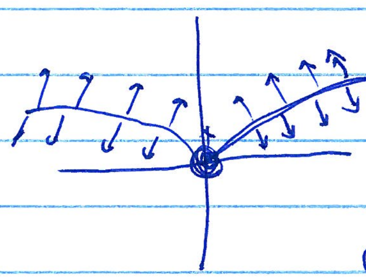
$SS(k_Z) \cap T_0^* M = Z^\circ := \{\xi;\ \langle v, \xi\rangle \geq 0\ \forall v \in Z\}$

<u>ex.</u> $Z =$



<u>ex:</u> $Z = \{x^2 = y^3\} \subset \mathbb{R}^2$





$SS(k_Z) \cap T_0^* M = T_0^* M$

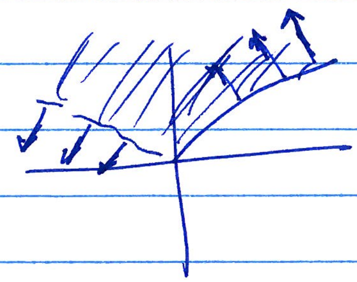
everything!

(not quite what we want)

$\Lambda = T^*_{Z_{reg}} M$ smooth (outside $(0,0)$)!

regular pts.

$Z =$



$SS(k_Z) = \Lambda^+ \cup Z$

$\Lambda^+$ = "positive half" of $\Lambda$.

In these examples, $SS(F)$ is Lagrangian (at regular pts.)

Question: $\Lambda \subset \dot{T}^* M$ smooth conic Lagrangian mfold.

remove 0 section

Find some (canonical) $F$ on $M$ s.t. $SS(F) \subseteq \Lambda$ on $M$.

$+ ( SS(F) \cap \dot{T}^* M = \Lambda )$

Case 1:

$\phi : \dot{T}^* M \times I \longrightarrow \dot{T}^* M$ (homog. hamiltonian isotopy)

$I$ open interval $\subset \mathbb{R}$, $0 \in I$.

$\phi_t = \phi_{\{T^*M \times \{t\}\}}$

$\phi_t$ sympl., homogeneous diffeo. isomorphism, $\phi_0 = id$.

$\Lambda' = \{(\phi(x,\xi), x, -\xi, t)\} \subset T^*(M \times M) \times I$

$\uparrow \qquad\qquad \uparrow$

$\Lambda \quad \longleftarrow \quad \dot{T}^*(M \times M \times I)$

$\exists$ $\Lambda$ conic, Lagrangian, and $\Lambda \simeq \Lambda'$!

$\Lambda = \{(—, \tau = f(\phi(x,\xi,t),t))\}$.

Thm: [-, k, S] $\exists!\ K \in D(k_{M\times M\times I})$ s.t.

- $SS(K) \subset \Lambda \cup (M\times M\times I)$

  $SS(K) \cap \dot{T}^*(M\times M\times I) = \Lambda$

- $K_0 = k_{\Delta_M}$ where $K_t = i_t^{-1}(K)$

  $i_t : M\times M\times\{t\} \hookrightarrow M\times M\times I$.

Moreover, supp $K \rightrightarrows M\times I$ are proper.

Case 2 $\quad M$ cpt. mfld

$\Lambda \subset T^*M$ exact Lagr. subm.

$\Lambda \xrightarrow{f} \mathbb{R} \quad df = i_\Lambda^*(\alpha_M)$.

$\tilde{\Lambda} = \{(x,t,\xi,\tau);\ (x, \xi/\tau) \in \Lambda\} \subset \dot{T}(M\times\mathbb{R}),\ \tau>0\}$.

Thm [Viterbo] Maslov class of $\Lambda = 0$.

$\exists F \in D^b(k_{M\times\mathbb{R}})$ s.t. $SS(F) \cap \dot{T}^*(M\times\mathbb{R}) = \tilde{\Lambda}$

$$F|_{M\times\{t\}} = \begin{cases} k_M & t \gg 0 \\ 0 & t \ll 0 \end{cases}$$

($\Rightarrow$ F-S-S, N-Z work).

Next time; Recover non-displaceability of 0-section, other symplectic results...

· behavior of SS(M) under sheaves operations
· proof of thm 1 ⇒ Arnold. · proof of thm 2.

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I) $f : M \to N$ morphism of manifolds

<u>direct image</u>: $F$, sheaf on $M$.

$f_* F$ on $N$ ; $(f_* F)(V) = F(f^{-1}(V))$

$V \subset N$

this is a sheaf.

$f_*$ is left exact $\rightsquigarrow Rf_* : D^b(k_M) \to D^b(k_N)$

$$T^*M \xleftarrow[f_d]{} M \times_N T^*N \xrightarrow[f_\pi]{} T^*N$$

$$\overset{\shortparallel}{(f')}$$

<u>Prop</u>: If $f|_{\operatorname{supp} F}$ proper, then

$$SS(Rf_* F) \subset f_\pi f_d^{-1}(SS(F))$$

$$\overset{\cap}{\left(\bigcup SS(R^i f_* F)\right)}.$$

<u>Rmk</u>: $F \in D^b(k_M)$

$$\underline{SS(F)} \subset \bigcup_i SS(H^i F)$$

<u>inverse image</u> $G$ sheaf on $N$

$f^{-1}G$ sheaf associated to presheaf

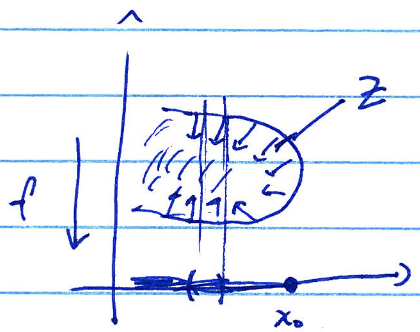
$$\left( \underset{\hat{M}}{U} \longmapsto \varinjlim_{V \ni f(U)} G(V) \right)$$

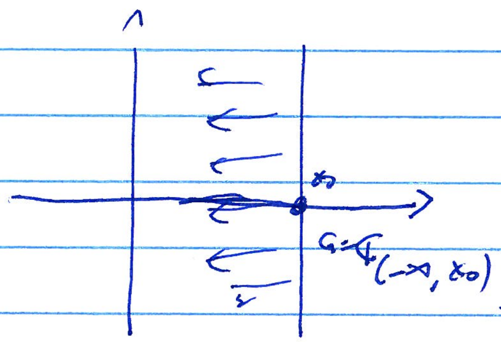
(problem that $f(U)$ is not open, so $G(f(U))$ not defined.

$f^{-1}$ exact functor, so induces

$$f^{-1}: D^b(k_M) \to D^b(k_N)$$

Prop: If $f$ is a submersion, then $SS(f^{-1}G) = f_d f_\pi^{-1}(SS(G))$




Prop: $p: M \times \mathbb{R}^n \to M$ projection.

assume $F \in D^b(k_{M\times\mathbb{R}^n})$.

if $SS(F) \subset (T^*M) \times \mathbb{R}^n$ ← zero section

Then $F \simeq p^{-1}(G)$ where $G = Rp_* F$.

tensor product

$F$, $G$ sheaves on $M$, $F \otimes G$ ~~assoc~~ sheaf associated to

$$U \longmapsto (F(U) \otimes_k G(U))$$

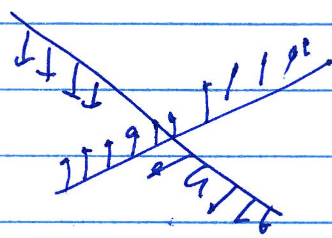
• $\otimes^L$ may be derived $\leftrightsquigarrow$ $F \otimes^L G$ for $F, G \in D^b(k_M)$

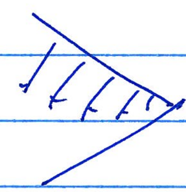
Prop: $F, G \in D^b(k_M)$

Assume $SS(F) \cap a(SS(G)) \subset M$.

$a(x,\xi) = (x, -\xi)$.

Then: $SS(F \otimes G) \subset SS(F) + SS(G)$



$k_Z \otimes k_{Z'}$ = 

$T_0^* M \cap SS(k_Z \otimes k_{Z'}) \subset$ 

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Composition of kernels:

$$\begin{tikzcd}
 & X \times Y \times Z \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\
X \times Y & X \times Z & Y \times Z
\end{tikzcd}$$

$K \in D^b(k_{X \times Y})$, $L \in D^b(k_{Y \times Z})$

$$K \circ L = Rq_{13!}(\underbrace{q_{12}^{-1} K \otimes q_{23}^{-1} L}_{H})$$

(if $q_{13}$ proper on supp $H$

$Rq_{13!}(H) = Rq_{13*}(H)$

$$\begin{array}{ccc} & T^*(X\times Y\times Z) & \\ \swarrow^{p_{12}} & \downarrow p_{13} & \searrow^{p_{23}} \\ T^*(X\times Y) & T^*(X\times Z) & T^*(Y\times Z) \end{array}$$

$A \subset T^*(X\times Y)$ $B\subset T^*(Y\times Z)$

$A\circ B = p_{13}(p_{12}^{-1}(A) \cap p_{23}^{-1}(B))$

e.g. if $A = \Gamma_\varphi$, $\varphi: T^*Y \to T^*X$

$B = \Gamma_\psi$, $\psi: T^*Z \to T^*Y$,

then

$A\circ B = \Gamma_{\varphi\circ\psi}$.

If $Z = \{pt\}$ $C\subset T^*Y$, $\Gamma_\varphi \circ C = \varphi(C)$

$\hookrightarrow$ our proposition gives conditions so that

$SS(K\circ L) \subset SS(K)\circ SS(L)$

proper

- $q_{13}|q_{12}^{-1}(\mathrm{supp}(K)) \cap q_{23}^{-1}(\mathrm{supp}\, L)$

- $p_{12}^{-1}(SS(K)) \cap (p_{23}^{-1}(SS(L))) \cap X\times T^*_Y Y\times Z$
  $\subset X\times Y\times Z$.

proof of thm 1 $\Rightarrow$ Arnold conj.

$\phi: \dot{T}^*M\times I \to \dot{T}^*M$ Hamiltonian isotopy

$N\subset M$ compact submanifold

$\psi: M\to \mathbb{R}$ $d\psi_x \neq 0$ $\forall x$ (M non-compact)

$$\Lambda_\psi = \{(x, d\psi_x)\} \subset \mathring{T}^*M$$

<u>Thm:</u> $\forall\, t \quad \phi_t(T_N^*M) \cap \Lambda_\psi \neq \emptyset.$

To recover ~~the Arn~~ Arnold's conjecture

$N$ compact

$M = N \times \mathbb{R}$

$\psi(x,t) = t.$

$\varphi: T^*N \times I \to T^*N$ Hamiltonian isotopy

$\varphi = \varphi_f \quad f: T^*N \to \mathbb{R}.$

$\phi: \mathring{T}^*M \times I \to \mathring{T}^*M$

$\phi = \phi_g$, where $\cancel{g(x,\xi,r,\tau) = \begin{cases} \tau f(x, \xi/\tau, t) & \tau \geq 0 \\ 0 & \tau \leq 0 \end{cases}}$

$$g(x,s,\xi,\sigma,t) = \begin{cases} \sigma f(x, \xi/\sigma, t) & \sigma \geq 0 \\ 0 & \sigma \leq 0 \end{cases}$$

assume $\varphi = \mathrm{id}$ outside a compact set.

$$\phi_t(T_N^* M) \cap \Lambda_\psi \overset{1-1}{\longleftrightarrow} \varphi_t(N) \cap N$$

proof: Let $K \in D^b(k_{M\times M\times I})$ s.t. $SS(K) \subset \Lambda \cup (M\times M\times I)$

$\Lambda \subset \dot{T}^*(M\times M\times I)$

$$\Lambda = \{ \phi_t(x,\xi), x, \xi, t, -f(\phi_t(x,\xi), t)) \} \quad \phi = \phi_f,$$

and $i_0^{-1} K = k_{\Delta_M}$

$i_t : M\times M\times \{t\} \hookrightarrow M\times M\times I.$

Set $F_0 = k_N \in D^b(k_M)$

$F := K \circ F_0 \in D^b(k_{M\times I})$

$$\begin{tikzcd} & I\times M\times M \arrow[dl] \arrow[dr] & \\ I\times M & & M \end{tikzcd}$$

$Z = pt.$

$$SS(F) \subset (\Lambda \circ T_N^* M) \cup (M\times I)$$

Since $SS(F_0) = T_N^* M$

$F_t := i_t^{-1}(F)$, $F_t = K_t \circ F_0 \in D^b(k_M)$

$SS(F_t) \subset \phi_t(T_N^* M) \cup M.$ (actually, $SS(F_t) \cap \dot{T}^* M = \phi_t(\dot{T}_N^* M)$)

We want to show $SS(F_t) \cap \Lambda_\psi \neq \phi.$

**Lemma:** $(\forall t, i)$ $H^i(M; F_t) \simeq H^i(M; F_0)$

pf:
$$\begin{tikzcd} M\times I \arrow[d, "q"] & F \arrow[d, mapsto] \\ I & Rq_*(F) \end{tikzcd}$$

supp $M \to M \times I$ both proper, so

supp $F \to I$ proper.

so base change formula: $H^i(M, F_t) \simeq (H^i Rq_* F)_t$.

but $Rq_* F$ is <u>constant</u>

it is enough to see that $SS(Rq_* F) \subset I$

in fact, $(\Lambda \circ T^*M) \cap M \times T^*I$

$\subset M \times I$.

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If the theorem is false, $\exists$ $t_0$ s.t. $SS(F_{t_0}) \cap \Delta_\psi \overset{(=?)}{\cancel{\neq}} \emptyset$.

By Morse lemma,

$$H^i(\psi^{-1}((-\infty, b)), F_{t_0}) \to H^i(\psi^{-1}((-\infty, a)); F_{t_0})$$

is an iso. $\forall$ $a \leq b$.

supp $F_{t_0}$ is compact, so we take $a < \psi(\operatorname{supp} F_{t_0}) < b$

then, $H^i(M; F_{t_0}) = 0$.