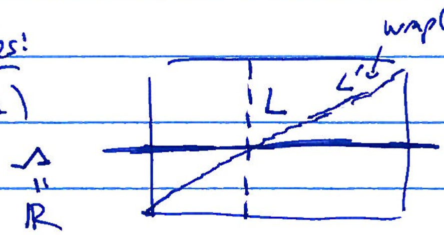
D. Nadler, Stable ∞-category of Lagrangian cobordisms.

Joint with H. Tanaka

Problem: understand (wrapped) Fukaya category $F_\Lambda(M)$ of exact target M.
with support Lagrangian $\Lambda \subset M$ (think of as a skeleton of M)

Examples:

1)



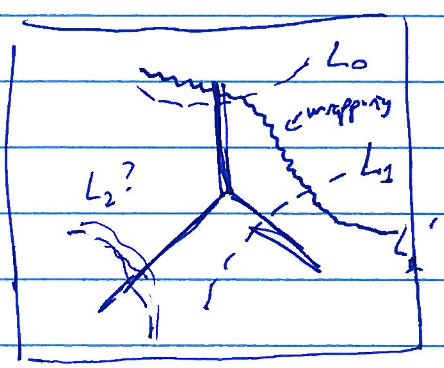
$M = T^*\mathbb{R}$

$F_\Lambda(M) = \mathbb{C}\text{-mod}$ (really, $D^b(\mathbb{C}\text{-mod})$, unbounded/infinite ?)

L represents $\mathbb{C}$, every other object is sums of L, etc.

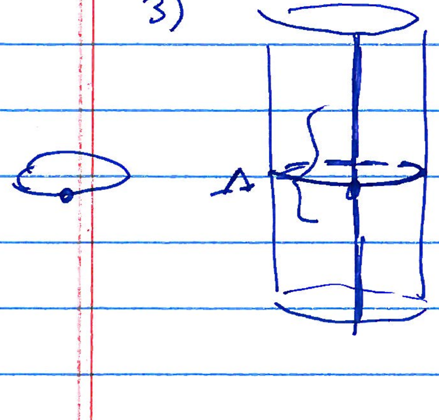
2)

$M = T^*\mathbb{R}$



$F_\Lambda(M) =$
$\mathbb{C}[\overset{L_0}{\bullet} \longrightarrow \overset{L_1}{\bullet}]$

3)



$F_\Lambda(M) = Sh_c($  $)$

Thm $M = T^*X$, $\Lambda = T^*_{\mathcal{S}}X$ $\quad \mathcal{S}$ = stratification of X

$$= \coprod_a T^*_{S_a} X$$

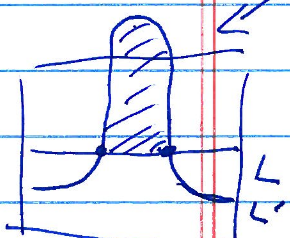
← conormal.

then $\mathcal{F}_\Lambda(M) \cong Sh_{\mathcal{S}}(X)$

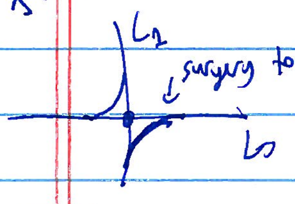
↑ enriched triangulated equivalence

$\mathcal{F}_\Lambda(M)$	$Sh_{\mathcal{S}}(X)$
analysis	topology
global	local
cones difficult	Cones tautological
flexible geometry	rigid algebra

ideal



cones in $\mathcal{F}_\Lambda(M)$



surgery to get cone, but lots of issues!

purely topological, no need for k, $\mathbb{C}$, etc.

<u>Main construction</u>: 1) stable ∞-category $Lag_\Lambda(M)$

$\mathbb{C}$ { (pre-triangulated) A_∞-category / — — dg category)

tame, closed, ...

<u>Objects</u>: $L \subset M$ exact lagrangian, $\pitchfork \Lambda$

Morphisms: Lagrangian cobordisms $P \subseteq M \times T^*\mathbb{R}_t$

2) ∞-functor

$$F: \mathrm{Lag}_\Lambda(M) \longrightarrow F_\Lambda(M)$$
$$L \longmapsto L$$

Conj: 1) F equiv. (after localization)

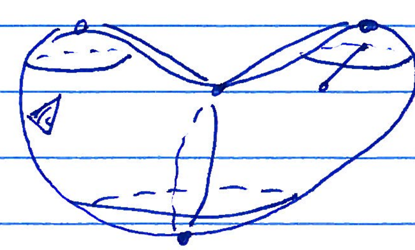
2) local

3 inspirations for morphisms:

1) Analogy:

$$\underbrace{\mathrm{Lag}_\Lambda(M) : F_\Lambda(M)}_{\text{category}} :: \overbrace{\text{Sing Chains}(X) : \text{Morse Complex}(X)}^{\text{Hom complex}},$$

Picture



Morphisms := "$\frac{\infty}{2}$ cycles" in Path(L_0, L_1)

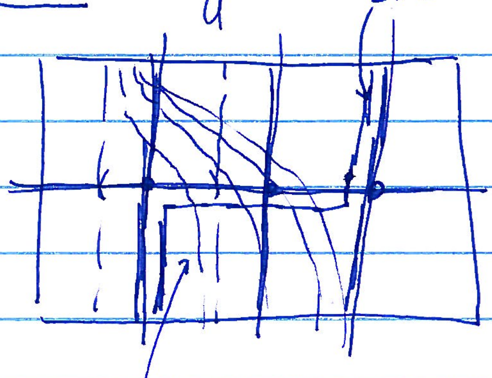
2) Noncharacteristic propagation

Motto: When are measurements invariant under motions?

Ex: $\mathcal{F}$ on $\mathbb{R}$, $\mathcal{U} \subset \mathbb{R} \rightsquigarrow \mathcal{F}(\mathcal{U})$

when does $\mathcal{F}$ stay same when we move $\mathcal{U}$?

Microlocal answer



Ans: $F(U)$ is invariant if $L_U \cap SS(F) = \emptyset$ near ∞ as L_U moves.

Idea: $Lag_\Lambda(M)$ is based on objects L, universal under noncharacteristic propagation.

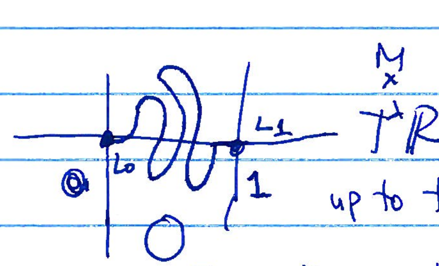
3) Pontryagin-Thom, TFT
Lurie ....

Rmk: $F_\Lambda(M)$ is difficult in part because
objs: Lag'ns. comps: disks.
Mor: $\cap$'s

Naive first try at $Lag_\Lambda(M)$:
objs: exact branes $L \subset M$.

morphisms:

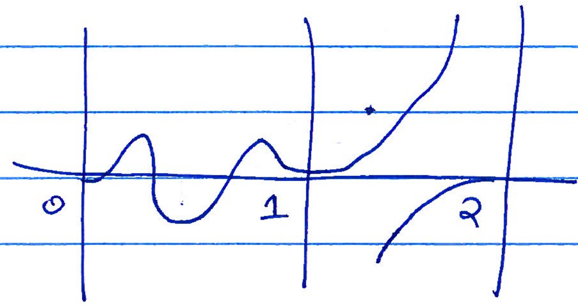
$P: L_0 \to L_1$.



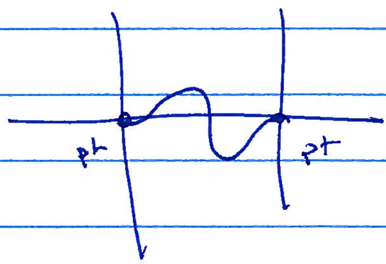
up to time reparam.

but, doesn't need to project to a curve.

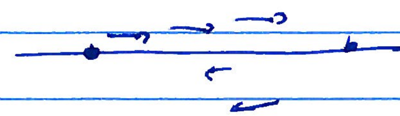
Comps:



Example: $M = \Lambda = pt$



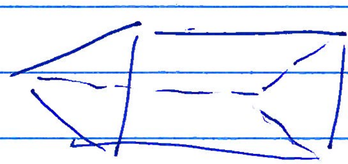
$pt \times T^*\mathbb{R} \rightsquigarrow$



vector field

Can enhance to ∞-category by taking families over Δ^k.

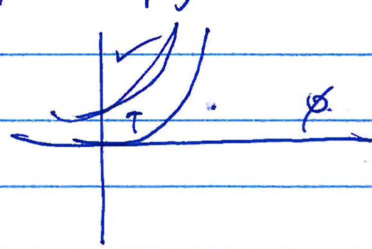
Ex. $M = \Lambda = pt$.

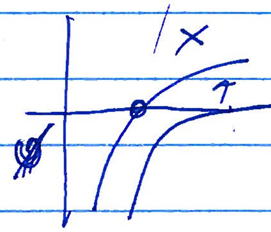


Δ^k $\longrightarrow \mathbb{R}_t$ "vector field"

$$T^*(M \times \mathbb{R}_t \times \Delta^k)$$

Prop: $\emptyset$ = empty brane is a zero object.





In fact: everything is zero.

Introduce Λ: $Lag^{\circ}_{\Lambda}(M)$ consist of Λ-noncharacteristic cobordisms

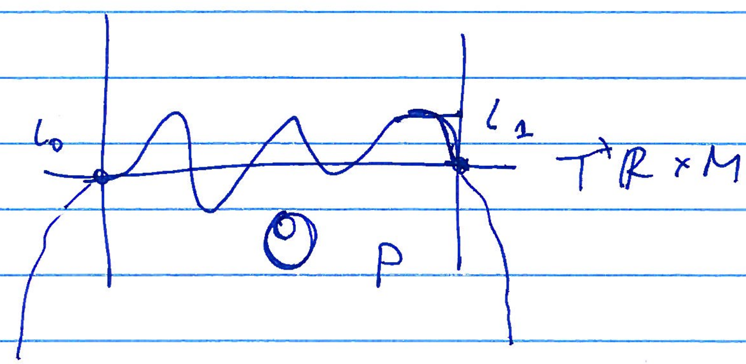
$$\exists\, \varepsilon > 0 \text{ s.t. } (p + \varepsilon\, dt) \cap (\Lambda \times \mathbb{R}) = \emptyset.$$

$Lag^{\circ}_{\Lambda}(M)$ interesting but difficult.

Take. $Lag_{\Lambda}(M) = \coprod_{k \to \infty} Lag^{\circ}_{\Lambda \times \mathbb{R}^k}(M \times T^*\mathbb{R}^k)$.

$$L \xmapsto[k]{} L \times T^*_{\{0\}}\mathbb{R}^k.$$

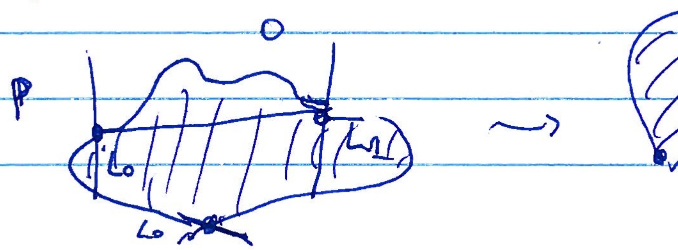
Cone $(P: L_0 \to L_1)$



$C(P) \subset M \times T^*\mathbb{R}$ by thinking of $T^*\mathbb{R}$ sideways, change time & space directions.

$Lag_{\Lambda}(M) \longrightarrow F_{\Lambda}(M)$

$L \longmapsto L$



Counts → gives invariant of P that lie in $\hom_F(L_0, L_1)$

sends id ↦ id.

Interesting even when $M, \Delta = pt.$

$M = \Delta = pt$ (so Lag in cobordisms is $T^*\mathbb{R}^n$), $Lag_{pt}(pt.)$

$\uparrow$ monoidal triangulated category b/c ($pt \times pt. = pt.$)

Conj: $L = pt.$ generates.

$L = pt.$ unit

Calculate: E_∞-spectrum.

$\Downarrow$

End($L = pt.$) is an abelian gp (spectrum)

Show: $\pi_0 = \mathbb{Z}$.

(actually $\mathbb{Z}/2$ but grading / rel. Pin structure gives you $\mathbb{Z}$).

End ($L = pt.$) comm. ring (E_∞).