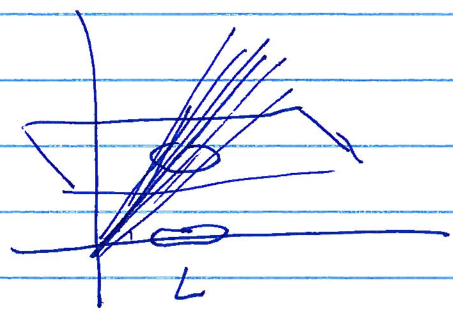
Tamarkin I

$L \subset T^* X$, $L \subset V$ any subset

$$\begin{array}{c|l} \text{Cone } L \subset V \times \mathbb{R} \ni t & \text{Cone } L := \\ \cup & \{(v,t) \mid t>0;\ \frac{v}{t} \in L\}. \\ V \times \mathbb{R}_{>0} & \end{array}$$



$T^*(X \times \mathbb{R}) = T^* X \times T^* \mathbb{R} \ni (t,k)$ $t \in \mathbb{R}$, $k \in \mathbb{R}$.
(x, ω)
$x \in X;\ \omega \in T^*_x X$.

$$L \subset T^* X \Rightarrow \text{Cone } L := \{(x, \omega, t, k) \mid k>0;\ (x, \tfrac{\omega}{k}) \in L\}$$
$$\subset T^*_{>0}(X \times \mathbb{R}) = \{(x, \omega, t, k) \mid k>0\}$$

$D(X \times \mathbb{R})$

$$T^*_{\leq 0}(X \times \mathbb{R}) \subset T^*(X \times \mathbb{R})$$

$$D_{\leq 0}(X \times \mathbb{R}) := \{\mathcal{F} \mid SS\,\mathcal{F} \subseteq T^*_{\leq 0}(X \times \mathbb{R})\}$$

Then define

$$D_{>0}(X \times \mathbb{R}) := \frac{D(X \times \mathbb{R})}{D_{\leq 0}(X \times \mathbb{R})}$$

$$\mathcal{F} \in D_{>0}(X \times \mathbb{R}) \Rightarrow SS\,\mathcal{F} \subset T^*_{>0}(X \times \mathbb{R})$$

Semi-orthogonal decomposition:

$$\mathcal{D}_{\le 0}(X,\mathbb{R}) \xrightarrow{I} \mathcal{D}(X\times\mathbb{R})$$

$\forall\, \mathcal{F} \in D(X\times\mathbb{R})$, can construct a distinguished triangle

$$\mathcal{F}_{>0} \to \mathcal{F} \to \mathcal{F}_{\le 0} \xrightarrow{+1}$$

$$\mathcal{F}_{\le 0} \in \mathcal{D}_{\le 0}\text{, and}$$

$$R\mathrm{Hom}(\mathcal{F}_{>0}, \mathcal{G}) = 0 \quad \forall\, \mathcal{G} \in \mathcal{D}_{\le 0}(X\times\mathbb{R}).$$

$$S \in D(\mathbb{R}\times\mathbb{R}) = \{A_{\{t_1\le t_2\}}\}.$$

Convolution functor

$$\circ : D(X\times\mathbb{R}) \times \mathcal{D}(\mathbb{R}\times\mathbb{R}) \to \mathcal{D}(X\times\mathbb{R})$$

$$\mathcal{F}_{>0} := \mathcal{F} * S.$$

Restriction: $A_{t_1\le t_2} \to A_{t_1 = t_2}$, inducing

$\mathcal{F} * S \to \mathcal{F} \circ A_{t_1 = t_2} = \mathcal{F}$, the map is

$$\mathcal{F}_{>0} \to \mathcal{F} \longrightarrow$$

Think of this category $\mathcal{D}_{>0}(X\times\mathbb{R}) \subset D(X\times\mathbb{R})$ as a full subcategory.

Typical example of an objects:

$$f : X \to \mathbb{R}.$$

Any local system

$$\mathcal{A}_{\{(x,t)\mid t\ge f(x)\}} \in \mathcal{D}_{>0}(X\times\mathbb{R})$$

Specific properties:

Given any $c \in \mathbb{R}$, can define translation

$$T_c : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$$
$$(x,t) \longmapsto (x, t+c)$$

Now given any $\mathcal{D}_{>0}(X \times \mathbb{R}) \ni \mathcal{F}$,
for $c > 0$, get a map

$$\mathcal{F} \longrightarrow T_{c*}\mathcal{F} \quad (\text{b/c } \mathcal{F} \sim \mathcal{F} * S).$$

Idea: cover symplectic neighborhoods by symplectic balls.

First, given $\qquad$ sympl. ball

$$T^*\mathbb{R}^N \supset B_R \longrightarrow T^*\mathbb{R}^N$$

associate an object in $\mathcal{D}_{>0}(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$

(Moreover, do for any family of embeddings).

First, a linear approximation: suppose ~~everything~~ all maps above linear.

$$A : Sp(2n) \times T^*\mathbb{R}^n \longrightarrow T^*\mathbb{R}^n.$$
α Liouville form

$$\rightsquigarrow L \subset T^*(Sp(2n) \times \mathbb{R}^n \times \mathbb{R}^n)$$

$$\downarrow \pi$$

$$\Gamma_A \in Sp(2n) \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n$$

$A^*\alpha =$
$p_2^*\alpha + \eta + dH$
$\eta \in \Omega^{1,0}(Sp(2n) \times T^*\mathbb{R}^n)\}$

$p_2 : Sp(2n) \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n.$

H Legendrian lift

$\rightsquigarrow$ get $\Lambda \subset T^*(Sp(2n)) \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times \mathbb{R}_t$

(exactly graph of H) $\uparrow$ contactization.

Legendrian submanifold (Rmk: if Lag. has an "anti-diagonal," can promote to Legendrian ??)

$\mathrm{Graph}(\eta) : Sp(2n) \times T^*\mathbb{R}^n \cancel{\times \mathbb{R}} \longrightarrow T^*Sp(2n)$

$\cap$

$L_\Lambda \subset T^*Sp(2n) \times T^*\mathbb{R}^n$

$L \subset T^*Sp(2n) \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n$

$Sp(2n) \times T^*\mathbb{R}^n \longrightarrow T^*\mathbb{R}^n.$

$\mathrm{Cone}\,\Lambda \subseteq T^*_{\geq 0}(Sp(2n) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R})$

Goal: look at

$D_{\geq 0}(Sp(2n) \times \mathbb{R}^n \times \mathbb{R}^n)_{\mathrm{Cone}\,\Lambda}$ micro-supported here.

Actually, pass to universal cover for simplicity:

$D_{\geq 0}(\widetilde{Sp(2n)} \times \mathbb{R}^n \times \mathbb{R}^n)_{\mathrm{Cone}\,\Lambda}$. $(\cong \mathcal{L}oc_{\widetilde{Sp}(2n)}\ \ )$

So look at $U \subset \widetilde{Sp}(2n)$ ball, any nhood of any point

$D_{\geq 0}(U \times -)_{\mathrm{Cone}\,\Lambda_U}$

$\cong$

$D(A\text{-mod})$

$\widetilde{Sp}$ simply connected & $H^2(\widetilde{Sp}) = 0$. So sheaf of categories (w/ local iso. w/ A-mod), dg has a global section

Consider $Q \in D_{>0}(\widetilde{Sp}(2n) \times \mathbb{R}^n \times \mathbb{R}^n)$

What may it look like?

restrict to the center ($\widetilde{Sp}(2n)$)

$$x \in \mathbb{Z} \longrightarrow \widetilde{Sp}(2n) \longrightarrow Sp(2n)$$

$$Q|_x \simeq A_{(x_1 = x_2,\, t \geq 0)}[2x],$$

One more thing:

rotate base, extends naturally

$$\begin{tikzcd} \otimes : GL(n) \arrow[r, phantom, "\subset"] \arrow[dr, "\substack{\text{actually lifts!}\\ \text{(need to check)}}"'] & Sp(2n) \\ & \widetilde{Sp}(2n) \arrow[u] \end{tikzcd}$$

So can restrict $Q|_{GL(N) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}}$

also have: $A((g, x, gx, t) \,|\, t \geq 0)$.

relation? $Q|_.$ differs from $A|_.$ by a twist:

$$Q \simeq A \otimes \underline{\mathcal{L}}$$

(on Fukaya-side, reflected by class that $w_2(L) = 0$, perhaps)

Want to make this into a (dg) monoidal category:

$$\widetilde{Sp}(2n) \times \underset{\substack{\shortparallel \\ X}}{\mathbb{R}^n} \times \underset{\substack{\shortparallel \\ X}}{\mathbb{R}^n} \times \mathbb{R}$$

$$= X \times X \times G$$

$$X = \mathbb{R}^n, \quad G = \widetilde{Sp}(2n).$$

alg-structure

$$Q \in \mathrm{Alg}\, \widetilde{\mathcal{D}}_{>0}(X \times X \times G) \text{ has a } \otimes.$$

Choose dg model for sheaves here, then can prove Q has A_∞ structure. (operad is very simple, so get essential uniqueness)

Now, do the following.

$$P : F \times B_R \longrightarrow T^*\mathbb{R}^n.$$

$$\forall f \in F: \quad \Gamma_F : B_R \underset{\substack{\text{sympl.} \\ \text{embedding.}}}{\hookrightarrow} T^*\mathbb{R}^n$$

Construct

$$\Lambda_P \subset \underbrace{T^*F \times B_R \times T^*\mathbb{R}^n}_{\Omega} \times \mathbb{R} \quad \text{Legendrian.}$$

what kind of sheaves can we associate here?

$$\Omega \subset T^*(F \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}) \text{ open subset}$$

Can define $\mathrm{Cone}\, \Lambda_P \subseteq \underbrace{\Omega \times T^*_{>0}}\, \mathbb{R} \subset T^*_{>0}(F \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$

$\bullet\ \underline{D_{>0}(F \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})}_{\text{Cone}\,\Delta_p}$

things supported away from Ω.

Good news: this is representable, but much harder to prove.

(more generally, given any conical subset, can define projector, which e.g. contains info about sympl. capacity)

$P: F \times B_R \longrightarrow \cancel{\text{Sympl}}\ T^*\mathbb{R}^n$

$$\begin{tikzcd} DP\big|_{F\times 0} : F \arrow[r] \arrow[dr, "\text{suppose? } \exists"'] & Sp(2n) \\ & \widetilde{Sp}(2n) \arrow[u] \end{tikzcd}$$

derived category of local systems

Then this category is $\sim$ ~~Dog~~ $Loc(F)$.

(need to reduce to linear case, by conjugation, by a factor dep. on λ, let $\lambda \to \infty$).

~~B_R~~