D. Auroux, HMS for hypersurfaces in toric varieties (or complete intersections) (joint w/ M. Abouzaid, much in progress) 11/17/2016

Geometric setup (Abouzaid-Auroux-Katzarkov):

$$H = \{ f_t(x_1, \dots, x_n) = \sum_{\substack{\alpha \in A \subseteq \mathbb{Z}^n \\ \text{finite}}} c_\alpha t^{\rho(\alpha)} x^\alpha = 0 \} \subset (\mathbb{C}^*)^n$$

hypersurface, $|t| \ll 1$ (near trop. limit)

$\rho : A \to \mathbb{R}$.

construct SYZ type mirror (Y, W)

Y toric CY $(n+1)$-fold, $\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^n \oplus \mathbb{R} \mid \eta \geq \operatorname{Trop} f(\xi_1, \dots, \xi_n)$

$$= \max_{\alpha \in A} (\langle \alpha, \xi \rangle - \rho(\alpha)))$$

different sign convention than usual.

$W_0 = -z^{(0, \dots, 0, 1)}$ (LG potential)

(vanishes to order 1 along toric divisors.)

Key ex:

$$\Pi_{n-1} = \{ 1 + x_1 + \dots + x_n = 0 \} \rightsquigarrow Y = \mathbb{C}^{n+1}$$
$$W_0 = -z_1 \cdots z_{n+1}$$

mirror construction uses:

$$\mathcal{F}(H) \hookrightarrow \mathcal{F}(X, W_X, s),$$

where $X = Bl_{H \times 0}((\mathbb{C}^*)^n \times \mathbb{C})$

$H = f^{-1}(0) \subset (\mathbb{C}^*)^n$

W_X = coord. function

Morse-Bott with crit. $\simeq H$.

($s \in H^2(X, \mathbb{Z}/2)$ some background class accounting for non-Spinness).

& now develop SYZ mirror symmetry for $(X, W_X, (s)) \longleftrightarrow (Y, W)$

* Partial compactification:

$$\begin{array}{ccc} H & \subset & (\mathbb{C}^*)^n \\ \cap & & \cap \\ \bar{H} & \subset & V \end{array}$$

toric Fano ($c_1 \geq 0$)

$f \in \mathcal{O}(\mathcal{L} \downarrow V)$;

then mirror to $\bar{H}$ is $(Y, W = -z^{(0,0,\dots,0,1)} + (\text{1 term per toric divisor of } V))$.

Ex: $\{1 + \Sigma x_i = 0\}$

$$\begin{array}{ccccc} \Pi_{n-1} & \subset & (\mathbb{C}^*)^n & \longleftrightarrow & (\mathbb{C}^{n+1}, -z_1 \cdots z_{n+1}) \\ \cap & & \cap & & \\ \mathbb{CP}^{n-1} & \subset & \mathbb{CP}^n & \longleftrightarrow & (\mathbb{C}^{n+1}, -z_1 \cdots z_{n+1} + Tz_1 + \dots + Tz_{n+1}). \end{array}$$

(stabilization of usual mirror).

(Complete intersections:

$$\overline{H}^{n-k} := f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0) \subset V^n;$$

mirror Y toric $(n+k)$-fold, $W = (k \text{ main terms}) + (1 \text{ term per divisor in } V)$!))

Two directions of HMS:

1) wrapped Fukaya $\mathcal{W}(H) \simeq D^b_{sing}(Y, W)$

cases done:

(+ punctured spheres, in 1D) AAEKO

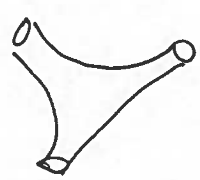
- Riemann surfaces $\subset (\mathbb{C}^*)^2$, H. Lee (via pair-of-pants decompositions).
- Higher pants: $\mathcal{W}$?? but see Nadler
- compact $\mathcal{F}$ of pants: Seidel, Sheridan ⇝ compactifications.

$$D^b Coh(Z) \xrightarrow{\text{quotient functor}}$$

Rmk: Expect: $\mathcal{W}(H) \xleftrightarrow{\sim} D^b Sing(W_0) = D^b Coh(Z)/Perf(Z)$

for $H \underset{hyp}{\subset} (\mathbb{C}^*)^n$ (or cases w/ "triv. normal bdle")

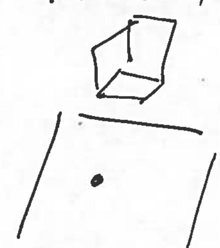
Ex:



$1 + x_1 + x_2 = 0$

$\longleftrightarrow$ $Y = \mathbb{C}^3$, $W_0 = -z_1 z_2 z_3$

$Z = W_0^{-1}(0)$

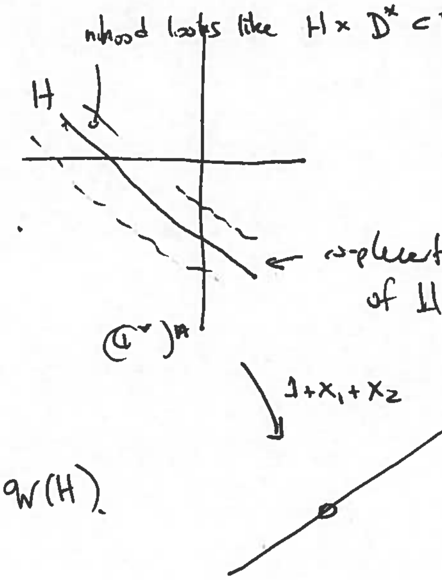


Expect:

$$\begin{array}{ccc} \mathcal{W}(H) & \xleftrightarrow{\simeq} & D^b Sing(W_0) \\ \uparrow \rho \text{ (understand this?)} & & \uparrow q \\ \mathcal{W}((\mathbb{C}^*)^n - H) & \longleftrightarrow & D^b Coh(Z) \xleftarrow{i_*} D^b Coh(Z_0). \end{array}$$

$FS((\mathbb{C}^*)^2, x_1 + x_2 + 1)$

$FS((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n - H)$



In general, $(\mathbb{C}^*)^n \setminus \Pi_{n-1} \simeq \Pi_n$

How to get ρ? Use: Viterbo restriction

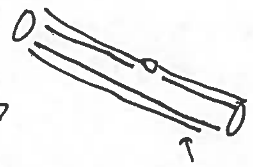
$$\mathcal{W}((\mathbb{C}^*)^n - H) \underset{[Ab\text{-}Se]}{\xrightarrow{\text{Viterbo}}} \mathcal{W}(H \times \mathbb{C}^*) \xrightarrow{\text{test against gen. point}} \mathcal{W}(H).$$

$$\Pi_2 = (\mathbb{C}^*)^2 \setminus \Pi_1 \longleftrightarrow (\mathbb{C}^4, -z_1 z_2 z_3 z_4)$$

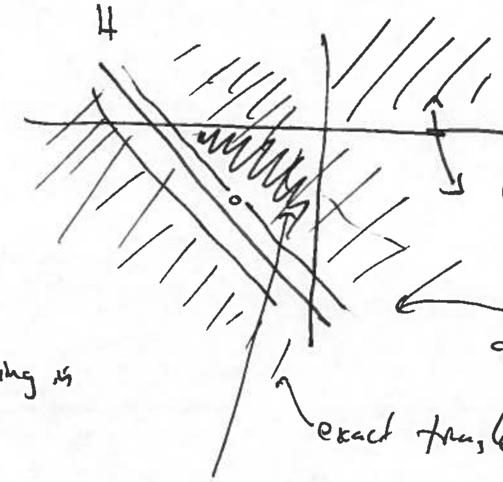
$$\longleftrightarrow \{z_1 z_2 z_3 = 0\} \subset \mathbb{C}^3 .$$

Knörrer periodicity

Ex: generates for



$\rightsquigarrow$



$\mathcal{O}_Z$; disjoint fm, so dies when restricting.

can realize exact triangles by pushing in copies of this.

exact triangle relation

FS Lag'n.

Rmk: need to be careful about Liouville structures...

What's Z_0? if $f = \sum_{\alpha \in A \subset \mathbb{Z}^n,\ 0 \in A} t^{\rho(\alpha)} x^{\alpha} = 0$; then

$$Z = W_0^{-1}(0) \subset Y$$

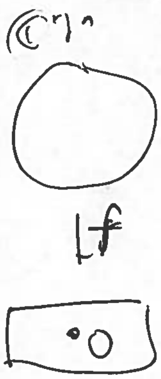
$$\text{''} \bigcup_{\alpha \in A} Z_\alpha .$$

Z_0 (plus 1 coordinates)

Have inclusion

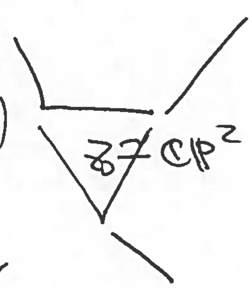
$$D^b(Z_0) \longrightarrow D^b(Z) \xrightarrow{q} D^b_{sing}(Z)$$

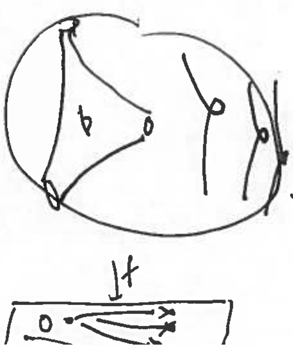
$(\mathbb{C}^*)^n$



other ex: $H: 1 + t x_1 + t x_2 + \frac{t}{x_1 x_2} = 0 \subset (\mathbb{C}^*)^2$

$$Y = \mathcal{O}(-3)$$
$$\downarrow$$
$$\mathbb{CP}^2$$





$Z = \bigcup$ toric components.

$$D^b(Z_0) \xrightarrow{i_*} D^b(Z) \xrightarrow{q} D^b_{sg}(Z) = D^b Coh(\mathcal{H})$$

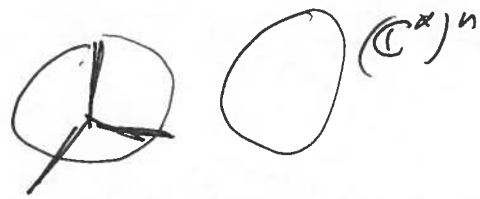
$$\updownarrow \qquad \downarrow$$

$$FS((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n \setminus H) \to \mathcal{W}(H)$$

(restriction from $\mathbb{CP}^2$ to anticanonical divisor)

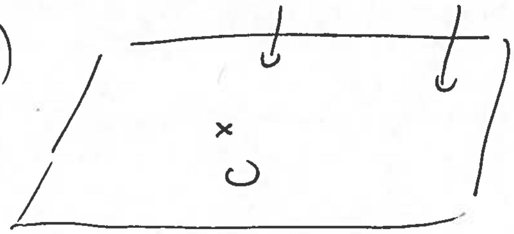
Other direction

$D^b Coh(H) \longleftrightarrow \mathcal{F}(Y, w)$.



First, $H \subset (\mathbb{C}^*)^n$. fibrewise wrapped.

$Coh(\Pi_{n-1}) \longleftrightarrow \mathcal{F}(\mathbb{C}^{n+2}, -z_1 \cdots z_{n+2})$



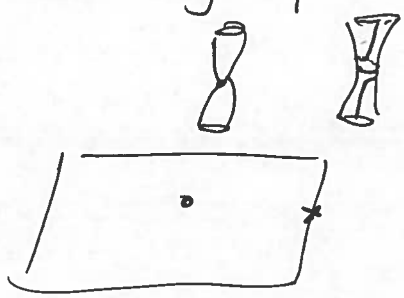
Features: • The generic fiber of W_0 is $\simeq (\mathbb{C}^*)^n$
(& zero fiber is toric degeneration)

• The monodromy around 0 fiber is "shear along Trop H".

Ex: $\mathbb{C}^2$

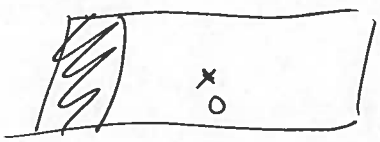
One-D. ~~dual~~ $-z_1 z_2 \downarrow$

$\mathbb{C}$



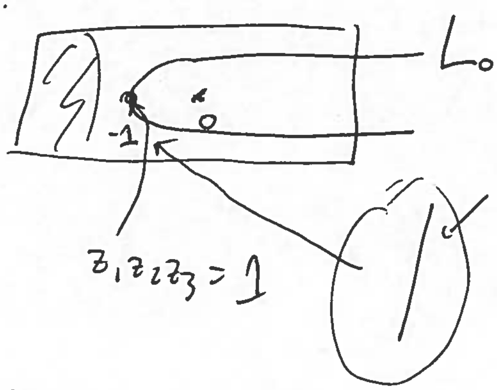
Dehn twist:
shift torus fibers by differing amounts, eventually π at 0 & $0/2\pi$ near ∞.

ob $\mathcal{F}(Y, w)$ should have Lag'ns w/ $w(L)$ outsides



morphisms ~~move~~ around Lag'ns.
(need to completely wrap in fibers).

Ex: arc:



$(\mathbb{C}^*)^2$; has a real structure & a real pos. locus

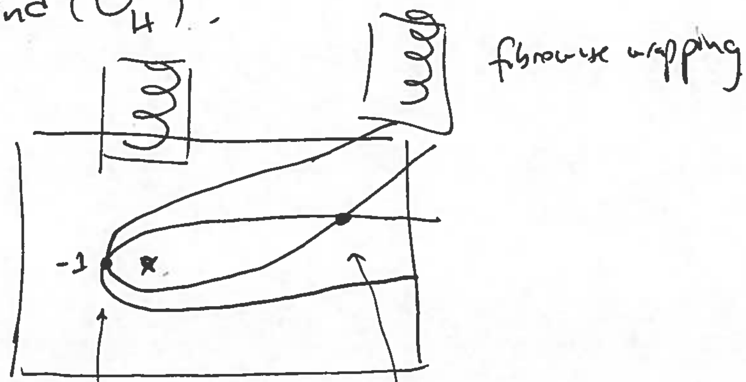
& transport it around along arc to get L_0.

(~~we~~ use toric Kähler form on Y, but would expect exact class to work
for technical simplicity at vertical ∞).

Claim: $\mathrm{End}(L_0) \simeq \overset{\mathbb{C} \text{ or } \Lambda \text{ depending}}{\mathbb{K}}[x_1^{\pm1},\dots,x_n^{\pm1}]/f$ as an algebra.

$$= \mathrm{End}(\mathcal{O}_H).$$

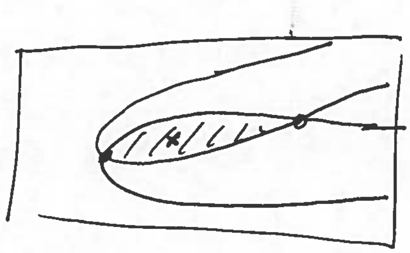
Sketch:



$\mathbb{K}[x_1^{\pm1},\dots,x_n^{\pm1}]$ ↺ usual alg. structure

$\mathbb{K}[x_1^{\pm1},\dots,x_n^{\pm1}]\theta$ — deg -1. & module over LHS.

$d = (-\cdot f)$

Differential: counts  using upgrade of [Seidel, Cho-Oh]

If $\bar{H} \subset V$ toric Fano, e.g.: consider

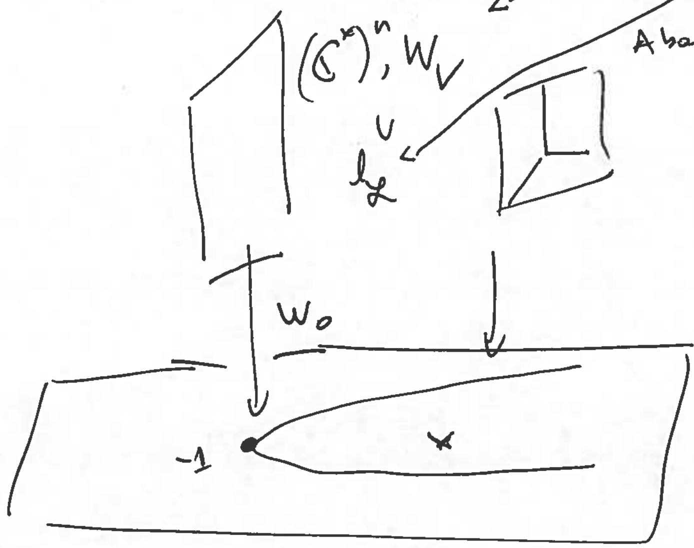
$$\mathbb{CP}^{n-1} \subset \mathbb{CP}^n \longleftrightarrow \Big(\mathbb{C}^{n+1},\ \underbrace{-z_1 \cdots z_{n+1}}_{W_0} + \underbrace{Tz_1 + \dots + Tz_{n+1}}_{W_V \text{ extra terms}}\Big).$$

Study $\mathcal{F}(Y, W_0 + W_V)$ by considering Lag'ns which are simultaneously admissible w.r.t. W_0 ~~and~~ and W_V (so if either W_0 or $W_V \to -\infty$, then L misses these regions; $\Rightarrow$ same for $W_0 + W_V$)

picture:

$(\mathbb{C}^*)^n, W_V \longleftrightarrow V. \longrightarrow \mathcal{L}$ line bdl.

Abouzaid mirror



Rmk: monodromy of W_0, in base "shear" along Trop(H) (which preserves admissibility); acting on in fiber

$$FS((\mathbb{C}^*)^n, W_V) \circlearrowleft \text{monodromy of } W_0$$

$$\Big\updownarrow \text{mirror}$$

$$- \otimes \mathcal{O}(H) \text{ on } D^b(V)!$$

Now, redo transport picture to get

$L_{\mathcal{L}}$: transport of $l_{\mathcal{L}}$, the Abouzaid mirror of $\mathcal{L}$.

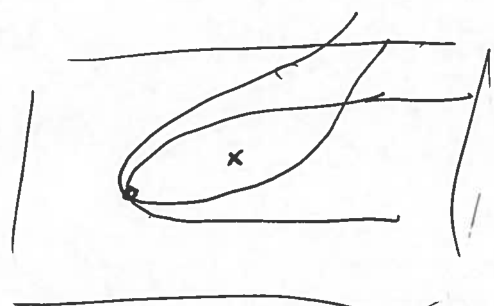
$$\& \ \hom(L_{\mathcal{L}}, L_{\mathcal{L}'})$$

$$\Big\updownarrow$$

$$\left(\hom(l_{\mathcal{L}}, l_{\mathcal{L}'}) \right.$$

$$\Big\uparrow -\circ f \text{ (mult. by defining eq'n.)}$$

$$\left. \hom(l_{\mathcal{L}\otimes\mathcal{O}(H)}, l_{\mathcal{L}'}) \right)$$



Complete intersections: iterate this, get koszul complex for calculating sheaf cohomology.