K. Fukaya, Monotone Lag'ns in divisor complement (work in progress w/ A. Daemi) 11/18/2016

(X, ω) symp. manifold, J. $D \subset X$ codim-2 sm. m'fold.

Assume: J is integrable in a nbhd of D, & $D \subset X$ complex submanifold.

Consider $L \subset X \setminus D$ Lag'n submanifold.

Def: $L \subset X \setminus D$ is monotone $\iff \exists\ c > 0$ s.t. $\forall\ (D^2, \partial D^2) \xrightarrow{u} (X \setminus D, L)$,

then $\int u^* \omega = c\mu(u)$ Maslov index.

(key point: only putting this assumption for disks outside D)

Thm: If $L_1, L_2 \subset X \setminus D$ are monotone & $\mu(u) > 2$ (all discs on either L_i?),

$\Rightarrow \exists\ HF(L_1, L_2; X \setminus D)$ satisfying:

$\mathbb{R}$-vec. space

(Rmk: don't know that it only depends on $X \setminus D$; may depend on D).

① Inv. of Ham. diffeos on $X \setminus D$

② If $L_1 \pitchfork L_2$, $\#(L_1 \cap L_2) \geq \operatorname{rk} HF^\cdot$

③ $\exists$ spectral seq. $H^\cdot(L) \Rightarrow HF(L, L; X \setminus D)$.

Notes:

① If $D = \emptyset$, then $\boxed{\text{Oh}}$

② If $X \setminus D$ is convex, then basically also

Main thing here is no assumptions about convexity of $X \setminus D$.

Possible generalizations

ⓐ D can be a normal crossings divisor.

ⓑ $G \circlearrowright X$ and D, L_i are G-invariant; then can do things equivariantly; $HF_G(L_1, L_2; X \setminus D)$.

ⓒ $\mu: X \longrightarrow \mathfrak{g}^*$ moment map, $L_1, L_2 \subset \mu^{-1}(0)$ G invariant (w/ G free on $\mu^{-1}(0)$. Just need bounding cochains (no need for bulk classes)).

Then, expect to show $HF_G(L_1; L_2; X \setminus D) \cong HF^*((L_1/G, b_1), (L_2/G, b_2); Y = X/G)$.

cook ups:

(d) Filtered A_∞ category $\mathcal{A}$, $\sigma(X \setminus D)$

in general, curved

objects (L, b) $L \subset X \setminus D$, where, if

L is monotone $\subset X \setminus D \Rightarrow L$ is unobstructed, & can take $b = 0$.

(e) $R = \Lambda_0 \langle\langle t_a\text{'s}, t_b\text{'s} \rangle\rangle$.

where: $a = 1, \ldots, m$ $\quad m = \mathrm{rk}\, H_*(D)$

$b = 1, \ldots, m'$ $\quad m' = \mathrm{rk}\, H_*(X \setminus D)$.

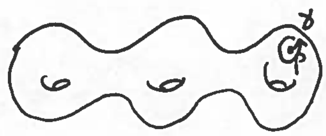
Can extend $Fuk(X \setminus D)$ to one over R, with $t = 0$ giving (1).

One application:

[Manolescu-Woodward]:

Take Σ_g, Riem. surface genus g, & G a cpct. semisimple Lie group.

Consider



the pair $\left\{ (\nabla, \xi) \;\middle|\; \begin{array}{l} \nabla \text{ flat connection on } \Sigma \setminus p \\ \text{w/ } hol_\gamma = \exp \xi \end{array} \right\} \Big/ \mathcal{G}_0$

gauge transf: g w/ $g(p) = id$.

$$// \quad \check{R}(\Sigma_g; G).$$

(Rmk: seems related to Geom. Langlands, but G cpct. group, not cplx $\to$ max. real part?)

[Huebschmann-Jeffrey]: $\check{R}(\Sigma, G)$ has a 2-form ω w/ $d\omega = 0$,

where ω is degenerate only on a small part.

$\mu : \check{R}(\Sigma_g, G) \longrightarrow \mathfrak{g}$ is a moment map for a G action, w/

ξ

$\check{R}(\Sigma, G) /\!/ G = R(\Sigma, G)$: moduli of flat G-bundles.

[Manolescu-Woodward: propose using to understand Lag'n Floer theory in or pertains to, e.g., Atiyah-Floer]

If $\partial H_g = \Sigma_g$ $\quad H_g$ is a handle body, this induces

$$\tilde{R}(H_g; G) = \mathrm{Hom}(\pi_1 H_g, G) \hookrightarrow \mu^{-1}(0) \subset R(\Sigma, G)$$

Lag'n submanifld, G-equiv. $\qquad$ G-equiv.

Also, [M-W] are a non-abelian symplectic cut:

$$\check{X} = \tilde{R}(\Sigma, G) \xrightarrow{\mu} \mathfrak{g} \xrightarrow[\text{Ad action}]{/G} W \quad \text{(Weyl chamber)}, \qquad \bar{\mu} : \check{X} \to W$$

Take a polygon $\subset W$ nbhd. of 0 (certain good polygons), call it Δ.

The non-ab. sympl. cut is: $\bar{\mu}^{-1}(\mathring{\Delta}) \cup \bar{\mu}^{-1}(\partial \Delta^{*}) / \sim \quad \Big\} =: X(\Delta)$.

[Woodward-Meinrenken]:

Ex: Say $G = SU(2)$

$$X \longrightarrow W = [0, \infty) \qquad \text{Consider } \varepsilon > 0, \text{ \& take}$$

$\bar{\mu}^{-1}([0, \varepsilon)) \cup \bar{\mu}^{-1}(\varepsilon) / S^1$

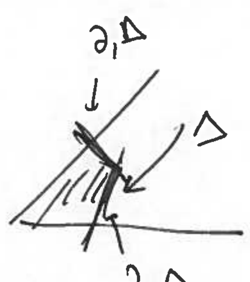
Have: $X(\Delta) = \mu^{-1}(\mathring{\Delta}) \cup \bar{\mu}^{-1}(\partial \Delta) / \sim$

$\qquad \cup$

$\qquad D = \bar{\mu}^{-1}(\partial \Delta)$

$\qquad \uparrow$ normal crossings divisor (smooth if $G = SU(2)$!

Ex: $G = SU(3)$

$W =$ 

Then, $X(\Delta) = \bar{\mu}^{-1}(\mathring{\Delta}) \cup \bar{\mu}^{-1}(\partial_1 \Delta^{*}) / S^1$

$\qquad \cup \bar{\mu}^{-1}(\partial_2 \Delta^{*}) / S^1 \cup \bar{\mu}^{-1}(\partial_1 \Delta \cap \partial_2 \Delta) / T$

Prove: $X(\Delta)$ smooth sympl. mfold (no degenerate locus for Δ, D small & D smooth n.c. divisor).

$\uparrow$ (sympl. form non-deg near ∂)

Back to $\check{X} = \check{R}(\Sigma, G) \xrightarrow{\mu} \mathfrak{g} \longrightarrow W \supseteq \Delta.$

If have $\Delta \subset W$ & get $X(\Delta)$.

Then, for a handle body H_g in fact

$$\check{R}(H_g, G) \overset{\text{Lagn.}}{\hookrightarrow} \mu^{-1}(0) \subset X(\Delta)$$

is <u>monotone</u> in $X(\Delta) \setminus \mathcal{D}$.

<u>Conj:</u> If $M^3 = H_g^1 \cup_{\Sigma_g} H_g^2$, then

$$HF_G^*(\check{R}(H_g^1, G), \check{R}(H_g^2, G); X(\Delta) \setminus \mathcal{D})$$

is an <u>invariant of the 3 manifold</u> M.

(If $G = SU(2)$, should give sympl. <u>side of Atiyah-Floer</u>).

[MW]: Case $G = SU(2)$. Then, considered:

$$X = R(\Sigma, SU(2)) \xrightarrow{\bar{\mu}} [0, \infty)$$

& $X(\tfrac{1}{2}) = \bar{\mu}^{-1}([0, \tfrac{1}{2})) \cup \bar{\mu}^{-1}(\tfrac{1}{2})/S^1$ is a <u>degenerate</u> sympl. manifold, but is <u>monotone</u> (without even removing $\mathcal{D}$). [MW] managed here to define $HF(\check{R}(H_g^1, SU(2)), \check{R}(H_g^2, SU(2)), X \setminus \mathcal{D})$

& suggest $\exists$ an $SU(2)$-equivariant version.

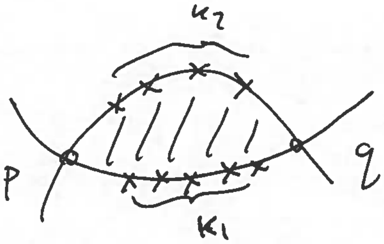
↑ in complement, using monotonicity in X.

Can also imagine generalization to toric CY's. (non-compact, non-convex; use fixed pt. localizations to <u>define</u> GW invts b/c not apriori defined;

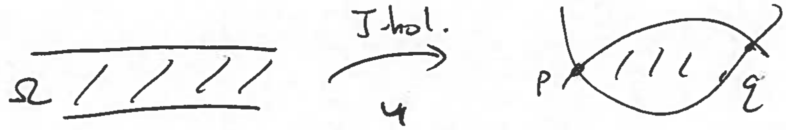
Can use this <u>same cut</u> trick to get a <u>cpt.</u> sympl. mfold X, w/ D, & $X \setminus D$ is CY

Sketch of proof:

Say have $L_1, L_2 \subset X \setminus D$, & consider



, or rather



w/ $u(\Omega) \subset X \setminus D$).

Gives $\overset{\circ}{\mathcal{M}}_{k_1,k_2}(p, q; X \setminus D, \beta)$, $\beta = [u]$ ← oncompactified.

Issue: how to compactify? If $X \setminus D$ convex, maximum principle + usual cptification.

Types of breaking: ~~Auxiliary~~ Another moduli spaces:

$u: (D^2, \partial D^2) \longrightarrow (X \setminus D, L) \rightsquigarrow \overset{\circ}{\mathcal{M}}_{k+1}(L, X \setminus D; \beta)$

$[u] = \beta$.

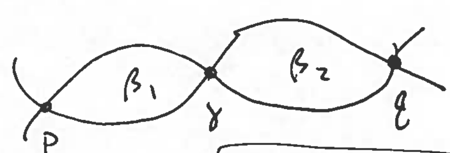
hol.

Main Lemma: Can compactify

$$\mathcal{M}_{k_1,k_2}(p, q; X \setminus D, \beta) \quad \& \quad \mathcal{M}_{k+1}(L; X \setminus D, \beta).$$

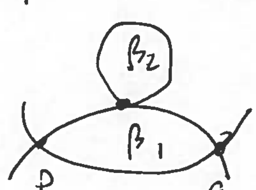
w/ $\partial \mathcal{M}_{00}(p, q, X \setminus D, \beta)$ has components:

(a) $\mathcal{M}_{00}(p, \gamma, X \setminus D, \beta_1) \times \mathcal{M}_{00}(\gamma, q, X \setminus D, \beta_2)$



w/ $\boxed{[\beta_i] \cdot D = 0}$

(b) $\mathcal{M}_{1,0}(p, q, X \setminus D, \beta_1) \times_L \mathcal{M}_1(L_1, \beta_2)$



w/ $\boxed{\beta_i \cdot [D] = 0}$

key point: usual cptification doesn't satisfy this.

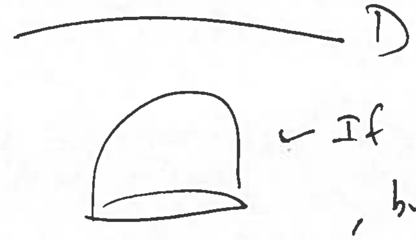
Note: as usual, Main Lemma $\xrightarrow[\text{Login Floer theo}]{\text{std. manipulations}}$ Main theorem.

Rmk: The usual stable maps compactification does not satisfy this!

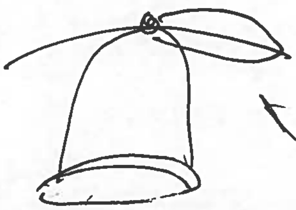
Need relative GW theory type compactification:

(J. Li, A. M. Li - Ruan, Ionel - Parker, B. Parker, Tehrani - Zinger).

Stable map compactification:

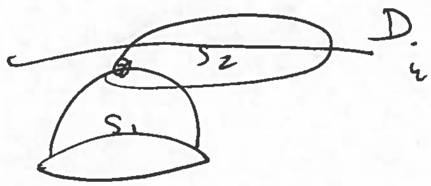


← If limit is smooth, won't touch D. (, but



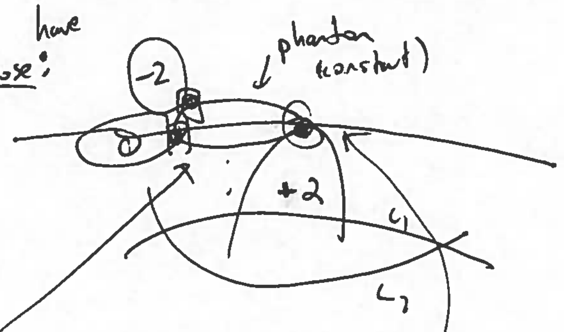
configurations may happen. If $D \cdot D > 0$, then this is also positive, so total $\cap$ w/ D is positive, ruling this out.

But can have:



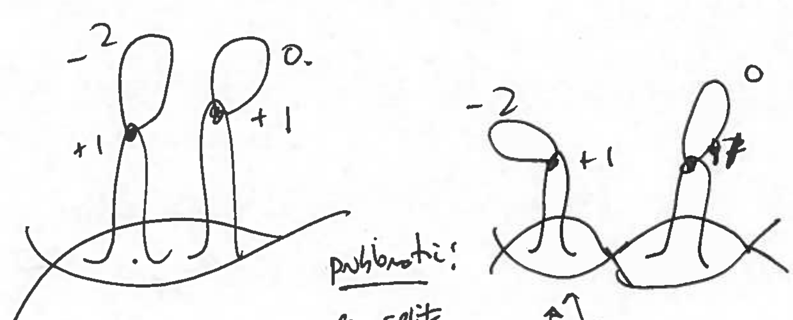
w/ $S_2 \cdot D < 0$, $S_1 \cdot D > 0$, so total $\cdot D = 0$

Suppose have:



D. Then, in total have intersection 0.

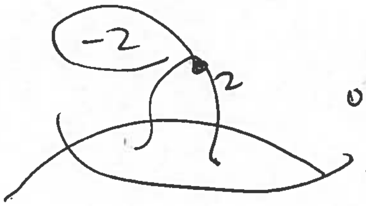
First, resolve singularity $\rightsquigarrow$ gives



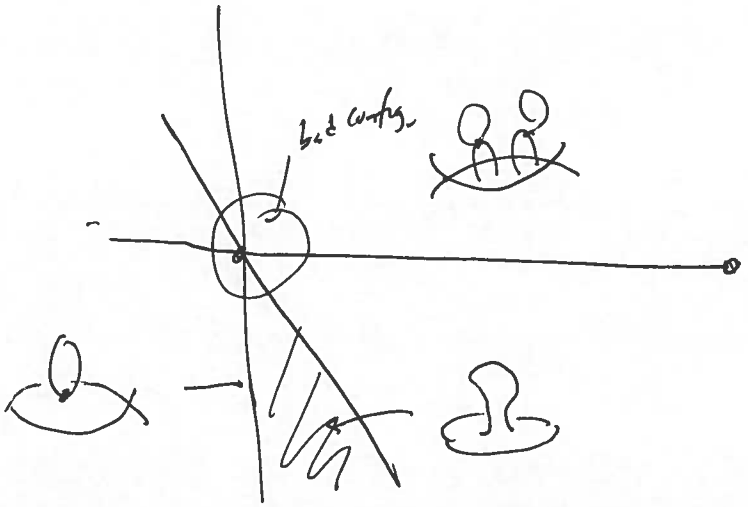
problematic: can split $\rightsquigarrow$ continue

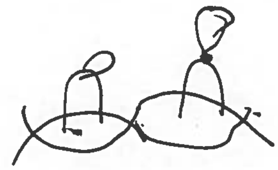
can never be a homology class in $X \setminus D$!

But, if resolve D; then get



ok!! So, funny thing: need to keep this & throw out





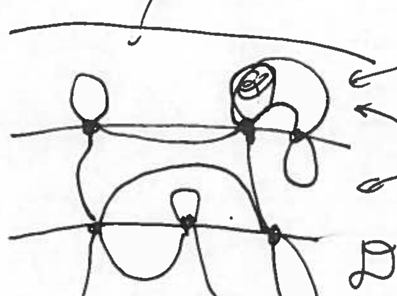
need to throw out, for monotonicity

need something here: away from divisor

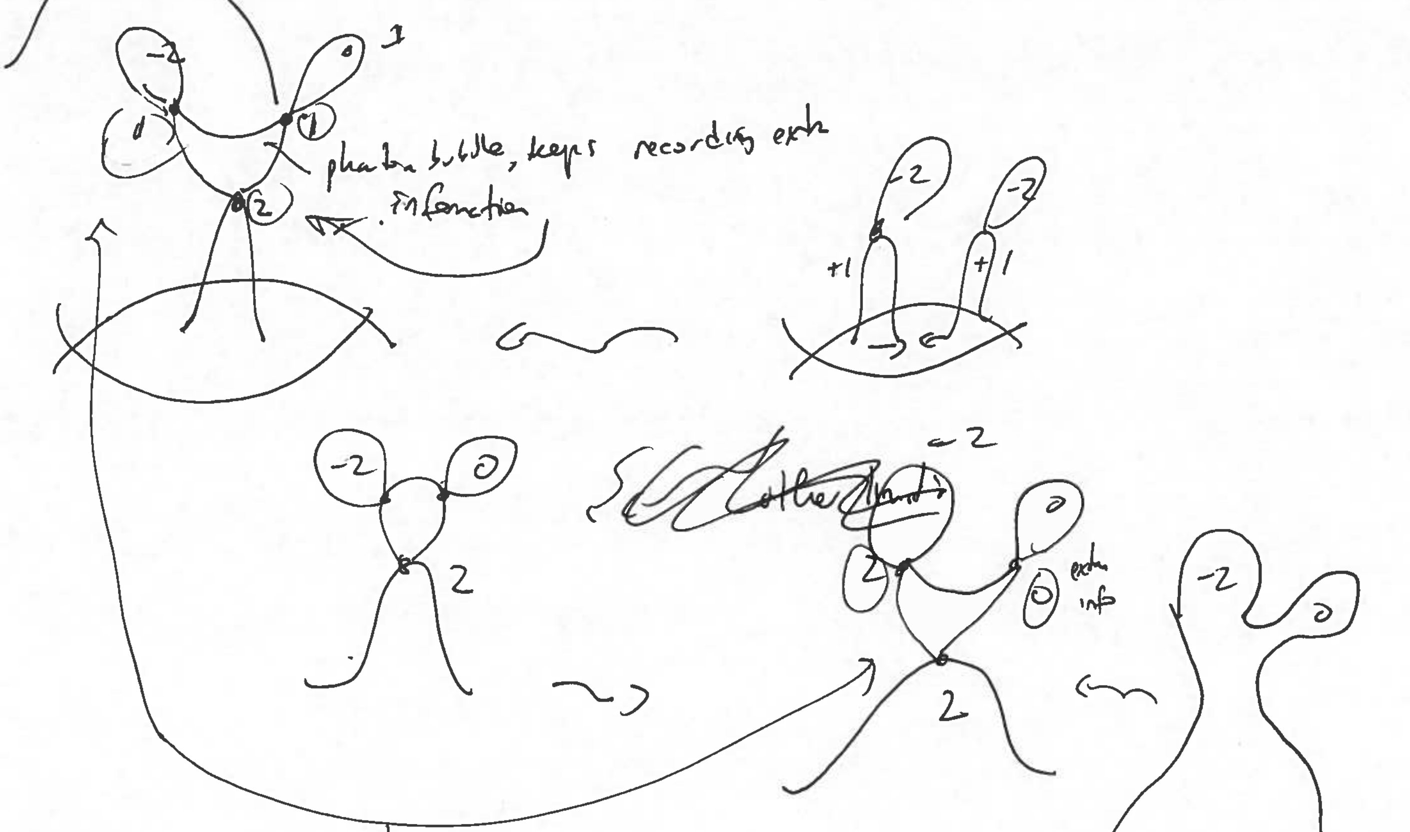
problem using usual stable map compactification

end away from 0 divisor.

something may come back (unlike SFT, where convexity forbids it).



Use this compactification instead:



don't agree:

so, can safely throw out LHS, which is good. Can also construct something close to SH here.