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Kuznetsov '03–'10

$X \hookrightarrow \mathbb{P}^{n+1}$ smooth hyp. degree d

w/ $D^b(X) = \langle \mathcal{A}_X, \mathcal{O}, \ldots, \mathcal{O}(m) \rangle$

$m = n+1-d$, & $\mathcal{A}_X$ is interesting.

$\mathcal{A}_X := \{ E \mid \mathrm{Hom}^\bullet(\mathcal{O}(k), E) = 0 \}$
$\forall k = 0, \ldots, m$.

Bondal-Orlov: $d < n+2$, $D^b(X) \simeq D^b(X') \iff X \simeq X'$

$\mathcal{A}_X \simeq \mathcal{A}_{X'} \not\Rightarrow X = X'$.

Serre functor: $S : \mathcal{T} \to \mathcal{T}$

$\mathrm{Hom}(E, F) = \mathrm{Hom}(F, SE)^*$

E.g., $\mathcal{T} = D^b(X)$, $S_X = (\omega_X \otimes \underline{\quad}) \cdot [n]$

Ex: $d = n+2$ $S_X \simeq [n]$ (Calabi-Yau)

[Kuznetsov:] $X \subset \mathbb{P}^{n+1}$, ~~deg~~ $\deg = d \leq n+2$.

$\sigma = (n+2)(d-2)$ $c = \gcd(d, n+2)$

$\Rightarrow S_{\mathcal{A}}^{d/c} \simeq [\sigma/c]$. $\Rightarrow$ $\mathcal{A}_X$ is fractional CY of dimension σ/d.

Remarks:

Observe if $X \subset \mathbb{P}^{n+1}$ is $\{f = 0\}$.

$\Rightarrow J(X) = \mathbb{C}[x_0, \ldots, x_{n+1}] / \{\partial_i f\}$ is Gorenstein of degree σ, e.g.

$J_k \times J_{\sigma - a} \longrightarrow J_\sigma \simeq \mathbb{C}$. perfect pairing.

<u>Ex:</u> $d = n+2 \rightsquigarrow$ CY dim n.

$d=3,\ n=4,\quad X \subset \mathbb{P}^5$ cubic.,

$\Rightarrow \mathcal{A}_X$ is CY2 (&, in a stronger sense, it's a "K3 category")

Have:

$$\begin{tikzcd} \mathcal{A}_X \arrow[r, hook, "i_*"] & D^b(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd}$$

w/ i^* left adjoint:

$$\mathrm{Hom}_{\mathcal{A}}(i^*E, F) \simeq \mathrm{Hom}_{D^b(X)}(E, (i_*)F) \quad \forall\ E \in D^b(X),\ F \in \mathcal{A}_X.$$

Can use this to define:

$$(1) : \mathcal{A}_X \longrightarrow \mathcal{A}_X, \quad E \longmapsto i^*(\underbrace{E \otimes \mathcal{O}(1)})$$

orthogonal to $\langle \mathcal{O}(1), \dots, \mathcal{O}(n)\rangle$ but <u>not</u> $\mathcal{O}$, so project

"degree shift functor" (comes from interpreting $\mathcal{A}_X$ as graded MF cat.).

<u>key lemma</u>: define $(d) := \underbrace{(1) \circ \dots \circ (1)}_{d \text{ times}}$.

Then, $(d) \overset{(*)}{\simeq} [2]$, if $d \leq \frac{(n+2)}{2}$.

<u>Corollary</u>: when $n=4,\ d=3 \Rightarrow \mathcal{A}_X$ is CY2.

<u>Proof</u>: $\mathrm{Hom}(i^*(E \otimes \mathcal{O}(3)), F) = \mathrm{Hom}(E \otimes \mathcal{O}(3), F)$

$$\underset{\text{on } X}{\overset{\text{Serre}}{=\!=}}_{\text{duality}} \mathrm{Hom}(F, E[4])^*$$

(slight cheating; need to show the (3) really is $i^*(E \otimes \mathcal{O}(3))$, not the translating (in))

$\Rightarrow S_{\mathcal{A}_X} \circ (3) = [4]$. Now $(*) \Rightarrow S_{\mathcal{A}_X} \simeq [2]$. ∎

Proof of ⊛ : have

$$\begin{tikzcd} \mathcal{A}_X(-m) \boxtimes \mathcal{A}_X \arrow[r, hook, "j"] & D^b(X \times X) \arrow[l, bend left, "j^*"] \end{tikzcd}.$$

(left adjoint

Now, start w/ any functor

$$Q \in D^b(X \times X) \rightsquigarrow j^* Q \quad \text{treat as a FM functor}$$

$$\begin{tikzcd} D^b(X) \arrow[r, "\Phi_{j^*Q}"] \arrow[d, dashed, "j^*"'] \arrow[dr, phantom, "\circlearrowleft"] & D^b(X) \\ \mathcal{A}_X \arrow[r] & \mathcal{A}_X \arrow[u, "i_*"'] \end{tikzcd}$$

"FM functors of $\mathcal{A}_X$"

the $(-m)$ discrepancy comes from shift by m in identifying Homs w/ tensor product

needed for FM.

E.g., $Q = \mathcal{O}_\Delta$

$$\rightsquigarrow P_0 = j^* Q \rightsquigarrow id_{\mathcal{A}_X}$$

$$Q = \mathcal{O}_\Delta(1) \rightsquigarrow P_1 = j^* Q \rightsquigarrow (1).$$

<u>Computation</u>: If $0 \le \ell \le d \le (n+2)/2$, then,

$$(⊛) \quad P_\ell = P_1 \circ \cdots \circ P_1 \rightsquigarrow (\ell)$$
$$\cong$$
$$j^* \mathcal{O}_\Delta(\ell).$$

$\Rightarrow$ kernel version of statement ⊛.