N. Sheridan, Versality of the relative Fukaya category 11/9/2016

X - cpct symplectic

$D \subset X$ divisor. Q: Understand $\mathcal{F}uk(X)$ from $\mathcal{F}uk(X \setminus D)$.

Examples:

$M \cong \mathbb{Z}^n$, $\Delta \subset M_{\mathbb{R}}$ reflexive polytope.

$\overline{\Sigma} \subset M^*_{\mathbb{R}}$ normal fan. $\overline{\Sigma}(1) =$ vertices of Δ° ← polar dual.

Σ = simplicial refinement ~~of~~ (of $\overline{\Sigma}$), with $\underset{\substack{\uparrow \\ \text{1-d rays}}}{\Sigma(1)} = \Xi := \left\{ \begin{array}{l} \text{lattice points in } \partial\Delta \text{ not} \\ \text{in the interior of a codim.-1} \\ \text{face} \end{array} \right\}$.

(these refinements always exist!)

$$(\Sigma, \Delta) \rightsquigarrow (Y, \mathcal{L}) \quad Y = \text{toric variety}, \ \mathcal{L} = \text{line bundle.}$$

Define $X = s^{-1}(0)$, $s \in \Gamma(\mathcal{L})$, and $D := X \cap \underbrace{\partial_{(toric)} Y}$, where
↑ toric divisor of Y.

$D_p = \bigcup_{p \in \Xi} D_p$ (assume X is smooth)

On the other side of mirror symmetry, take

$\overline{\Sigma}^\circ, \Delta^\circ$ ← polar duals

$(\overline{Y}^\circ, \mathcal{L}^\circ) \rightsquigarrow \mathcal{X}^\circ \subset \overline{Y}^\circ \times \mathbb{A}^\Xi$ ← family of hypersurfaces; ↙ coords. indexed by Ξ.

$$\downarrow$$

$\mathbb{A}^\Xi$. How to define $\mathcal{X}^\circ$?

Well, $\Gamma(\mathcal{L}^\circ)$ has a basis $\{\chi^p\}$ indexed by $\Delta^\circ \cap M^*$ ← lattice points in Δ°, and define

$$\mathcal{X}^\circ = \{\chi^\circ = \sum_{p \in \Xi} r_p \chi^p\}$$

r_p ← coord. in p factor of $\mathbb{A}^\Xi$.

χ° ↑ section corresp. $0 \in \Delta^\circ$.

Assume:

- X is smooth, $\omega =$ restriction of toric Kähler form from Y.

$$\Rightarrow [\omega] = \sum_p \lambda_p [D_p],$$

(assume each $\lambda_p > 0$, $\forall p$).

[Rmk: this is the Batyrev construction. when Δ is a simplex, due earlier to Greene-Plesser].

Thm (Smith-S.): Suppose Δ is a simplex, ω as before. Then $\exists\ \Lambda$-point $d(\lambda) \in \mathbb{A}^{\Xi}$, and Λ-linear quasi-equivalence

$$D^\pi \mathrm{Fuk}(X,\omega) \simeq D^b \mathrm{Coh}\left(X^\circ_{d(\lambda)}\right),$$

variety over Λ.

where $\mathrm{val}(d(\lambda)) = \lambda \in \mathbb{R}^{\Xi}$

gives a bit of information about what this Λ-point is. (leading order).

Recall: $\Lambda := \{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} : c_i \in \mathbb{C},\ \lim_{i \to +\infty} \lambda_i = +\infty \}$

$\mathrm{val} : \Lambda \longrightarrow \mathbb{R} \cup \{\infty\}$.

Let:

$X =$ cplx. compact manifold, $D \subset X$ snc divisor (simple normal crossings).

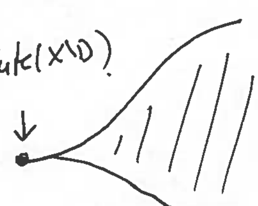
A relative kähler form is a kähler form ω equipped with $h : X \setminus D \longrightarrow \mathbb{R}$ kähler potential (w/ a prescribed form near D).

$\rightsquigarrow X \setminus D$ becomes a convex exact symplectic manifold. (has to do w/ prescribed form near D)

($\alpha :=$ primitive induced by X).

pictures [Seidel, 2002]

(*) $\mathrm{Fuk}(X \setminus D)$



$\mathcal{M}_{\text{käh}}(X,D)$ space

cptify by allowing ω to be infinite near strata of D.

Try to understand $\mathrm{Fuk}(X \setminus D)$ over $\mathcal{M}_{\text{käh}}(X,D)$ by understanding small nbhood of limit point (*), partially via a versality statement.

+ Rel Kähler form ω defines

$$[\omega] \in H^2(X, X\setminus D; \mathbb{R}) \cong \mathbb{R}^P, \quad D = \bigcup_{p \in P} D_p.$$

Def: $\exists$ open convex cone: $\mathrm{Amp}(X, D) \subset H^2(X, X\setminus D; \mathbb{R})$

$\quad$ "

$\quad$ effective ample divisors $\sum_p \lambda_p D_p$ supported on D.

Lemma: $[\omega] \in H^2(X, X\setminus D; \mathbb{R})$ is represented by a rel. Kähler form ω

$$\iff [\omega] \in \mathrm{Amp}(X, D).$$

Def: Let $N \subset \mathrm{Amp}(X, D)$ be an open convex (sub-) cone.

Define $NE(N) := \{ u \in H_2(X, X\setminus D; \mathbb{R}) \mid u \cdot a \geq 0 \ \forall a \in N \}$; monoid.

and define $R(N) := \mathbb{C}[\![NE(N)]\!]$

group ring, completed at the unique maximal toric ideal.

Note: There exists a homomorphism $R(N) \to \Lambda$, $r^u \longmapsto T^{\omega(u)}$ so long as $[\omega] \in N$.

"ω as a coh. class"

$= \omega(u) - \alpha(\partial u)$

Liouville form induced by α primitive.

(If $[\omega] \notin N$, this map won't converge).

Def'n: $\mathrm{Fuk}(X, D, N)$ has:

- Obj = closed exact Lag. branes ~~...~~ $L \in X \setminus D$. (to say "Lag'n", need to fix choice of ω, α once & for all. Might hope as move $\omega \in H^2(X, X\setminus D; \mathbb{R})$, can't make L's ___).
- $\hom(L_0, L_1) := R(N)\langle L_0 \cap L_1 \rangle$
- μ^* counts $u : (\mathbb{D}, \partial \mathbb{D}) \to (X, L_i)$ weighted by $r^u \in R(N)$.

homology class of disc u

Rmk: need to check answer really lies in $R(N)$; has to do with positivity of intersections (need to be careful if divisor is singular / $\exists$ a bubble tree in the divisor).

Compare: $\mathrm{Fuk}(X,\omega)$ has

ob := closed $L \subset X$

$\hom(L_0, L_1) := \Lambda\langle L_0 \cap L_1 \rangle$, μ^* counts u, weighted by $T^{\omega(u)}$.

$\rightsquigarrow$ get: $\mathrm{Fuk}(X, D, N) \otimes_{R(N)} \Lambda \longrightarrow \mathrm{Fuk}(X,\omega)$.

$$\downarrow$$

$$\operatorname{Spec} R(N) = \overline{\mathcal{M}}_{\text{käh}(X,D)}$$

The map from $R(N)$ to Λ is a "Λ point in $\overline{\mathcal{M}}_{\text{käh}(X,D)}$. Tensoring w/ Λ is "taking fiber" of family.

"The fiber of $\mathrm{Fuk}(X, D, N)$ at the Λ point $R(N) \to \Lambda$ is $\mathrm{Fuk}(X,\omega)$."

Also have $\mathbb{C}$-point $0 \in \operatorname{Spec} R(N)$;

fibre over this point is $\mathrm{Fuk}(X \setminus D)$.

$\Rightarrow$ $\mathrm{Fuk}(X, D, N)$ is a <u>deformation</u> of $\mathrm{Fuk}(X \setminus D)$ over $R(N)$.

(Rmk: note hom spaces are <u>flat</u> by ~~deformation~~ definition).

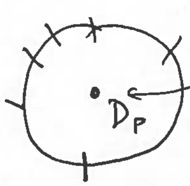
$\mathcal{A}$ is a <u>deformation</u> of $\mathcal{A}_0$ over $R(N)$ if $\mathcal{A} = \mathcal{A}_0 \otimes_{\mathbb{C}} R(N)$. (on level of morphism spaces),

with
$$\mu^* = \mu_0^* + \sum_{\substack{u \neq 0 \\ u \in NE(N)}} r^u \mu_u^* ;$$

Let $u_p \cdot D_q = \delta_{pq}$. Under (minor) hypotheses on N,

"u_p are the primitive small discs through D_p once.."

$\mu^*_{u_p}$ defines class $[\mu^*_{u_p}] \in HH^*(\mathcal{A}_0)$. (first <u>order deformation classes</u>)

count  intersect D_p once, no other divisors. c.f. CO.

We say $\mathcal{A}$ is a versal deformation if, for any other deformation $\mathcal{B}$ with same 1^{st} order def. classes, $\exists\ \psi^* : R(N) \xrightarrow[\text{automorphism}]{\sim} R(N)$

and $\mathcal{B} \cong \psi^* \mathcal{A}$.

Thm (S.): Let $\mathcal{A} \subset \mathrm{Fuk}(X, D, N)$, a deformation of $\mathcal{A}_0 \subset \mathrm{Fuk}(X \setminus D)$,

satisfies: (1) N 'nice'

(2) X is Calabi-Yau (or $c_1(X)$ torsion; but $\exists$ a version if X semipositive too).

(3) $H^2(X \setminus D) \cong 0$ (but, we'll see how to remove this next).

(4) $CO : SH^{\bullet}(X \setminus D) \longrightarrow HH^{\bullet}(\mathcal{A}_0)$ surjective

$\Rightarrow \mathcal{A}$ versal.

SH^2 classes roughly computed explicitly; in S. thesis, some computed, & also have to compute $H^2(X \setminus D) \to SH^2(X \setminus D) \xrightarrow[CO]{} HH^{\bullet}$, & use fact CO is an alg. homomorphisms.

Rmk:

Why? Def. theory appears in $HH^2 = \mathrm{im}\, SH^2$ (by assumption (4)).

(3) $\Rightarrow SH^2 = \mathrm{span}($linking loops around divisor$)$, which corresp. under CO to first order deformation classes. (4) $\Rightarrow HH^2$ gen by boundary divisor classes..

Now, any deformation in those directions, we can correct by fiddling w/ ψ^*.

If SH^2 has $H^2(X \setminus D)$ classes; some deformations of $\mathcal{A}$ don't come from geometric deformations.

Rmk: to eliminate (3):

(If $G \curvearrowright X, D, \mathcal{A}$, suffices $H^2(X \setminus D)^G = 0$)

Many cases: $G = \mathbb{Z}/2$, acting by antisymplectic involution; gets rid of lots of $H^2(X \setminus D)$.

Condition (1) is a real restriction, however; but:

Rmk: If $X \subset Y$ = toric variety, $D = X \cap \partial Y$, then any ω restricted ~~coming~~ from Y is nice.

To prove HMS: look for

$\mathcal{B} \subset \mathrm{Perf}(\mathfrak{X}^\circ)$ with

- $\mathcal{A}_0 \cong \mathcal{B}_0$
- 1st-order def. classes match.
- $\mathcal{A} \simeq \Psi^* \mathcal{B}$ by versality.

Then, base change, & apply automatic split-generation results (using fact that base changed $\mathcal{B}$ split-generates)

&, can also determine Ψ uniquely, using HMS $\Rightarrow$ Hodge MS.