Ivan Smith, Mirror Symmetry for the mirror quartic & other stories (joint work w/ N. Sheridan). 11/9/2016

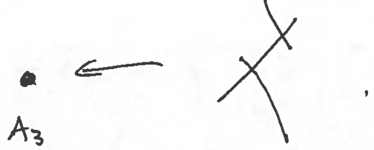
Mirror quartic

$\{x_0^4 + \cdots + x_3^4 + \lambda x_0 x_1 \cdots x_3\} \subset \mathbb{P}^3 \times \mathbb{C}_\lambda$, has action of

the group $\Pi = \mathbb{Z}/4 \times \mathbb{Z}/4 = \ker\left((\mathbb{Z}/4)^4/_{\mathbb{Z}/4} \xrightarrow{+} \mathbb{Z}/4 \right)$.

General fiber after dividing by Π has 6 A_3 singularities.

Resolve these



This is the mirror quartic $\check{X}$ contains 18 exceptional curves $[E_i]$ & hyperplane class H

Greene-Plesser HMS story:

$$D^\pi \mathcal{F}(X,\omega) \simeq D^b Coh\left(X^\circ_{d(\omega)}\right)$$

toric kähler form for $X \subseteq \mathbb{P}_\Delta$. quartic K3 surface $/\Lambda$) defined by explicit monomial eq'n whose coeffs. valuations encode data of $[\omega]$.

Thm (Sheridan, S.): Let $[\omega]$ be irrational toric kähler form. (coeffs. of $[D_i]$ lin. indep. over $\mathbb{Q}$.) then,

(a) the symplectic Torelli group $\ker(\pi_0 Symp(X,\omega) \longrightarrow \pi_0 Diff(X))$ is infinitely generated.

(b) If $S \subseteq X$ is a Lagrangian sphere, then the Dehn twist τ_S has no cube root in $\pi_0(Symp)$ (contrast smooth case where $\tau_S^2 \underset{smooth}{\simeq} id$, $\tau_S^3 \underset{sm.}{\simeq} \tau_S$.)

(c) If $L \subseteq (X,\omega)$ is a Lag'n torus of Maslov class zero, then $[L] \in H_2(X,\mathbb{Z})$ is non-zero (primitive)..

Rmk: (c) follows as always in footsteps of Seidel.

Standing on shoulders of Bayer & Bridgeland:

Let X° be a K3 surface $/\mathbb{C}$, There's a non-empty complex manifold

$\mathrm{Stab}(X^\circ) = \mathrm{Stab}(D^b(X^\circ))$: a point of $\mathrm{Stab}(X^\circ)$ comprises

- $Z : K_{num}(D^b(X^\circ)) \to \mathbb{C}$, (for us, $K_{num} \cong N(X^\circ)_{\mathbb{C}}$, $N(X^\circ) = H^0_{\mathbb{Z}} \oplus H^4_{\mathbb{Z}} \oplus \mathrm{Pic}(X^\circ)$).

  ↑ "numerical Groth. group": K_0/"kernel of Euler form":

— collection of subcategories $\mathcal{P}(\phi)$ of "ϕ-semistable objects"

s.t. every object has a HN-filtration

$$\begin{tikzcd}
0 \arrow[r] & E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_n = E \arrow[dl] \\
& A_1 \arrow[u] & & A_n \arrow[u] &
\end{tikzcd}$$

$0 \neq E \in D$ (cat.), $A_i \in \mathcal{P}(\phi_i)$, $\phi_1 > \cdots > \phi_n$.

- other axioms -

Highly non-obvious any exist; once $\mathrm{Stab}(D^b(X^\circ))$ is non-empty, $\mathrm{Aut}(D)$ acts on it.

$\mathrm{Stab}(D^b(X))$ non-empty

Using this, Bayer-Bridgeland prove: Suppose $\mathrm{Pic}(X^\circ) = \mathbb{Z} \cdot H$, Picard rank 1

Then, $\mathrm{Aut}(D^b(X))$ fits into an exact sequence:

$$1 \to \mathrm{Aut}_0(D) \to \mathrm{Aut}_{CY}(D) \to \mathrm{Aut}$$

↑ 'CY-equivalences' (autoequivalences preserving duality isomorphisms)

identified w/ π_1 (or orbifold π_1) of an explicit period domain.

If X° is a quartic K3 of Picard rank one, $\mathrm{Pic}(X^\circ) = \langle 4 \rangle$, the BB sequence.

becomes

equivalences of lattice containing $H^0 \oplus H^4$ w/ Pic lattice.

$$1 \longrightarrow \mathbb{F}^\infty \longrightarrow \mathbb{Z} * \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 * \mathbb{Z}/4 \longrightarrow 1$$

$\mathbb{F}^\infty$: π_1 (upper half plane minus infinite set)

$\mathbb{Z} * \mathbb{Z}/4$: π_1 (orbifold $\mathbb{P}^1$ parametrized by $\lambda \in$ previous explicit family),

[BB]: If E is semirigid, meaning $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \cong \mathbb{C}^2$, (not always even rank by duality reasons; so this is next case from 0).

(**) then $\exists$ a stability condition s.t. $\mathcal{E}$ is semistable.

Rmks:

(i) Via symplectic monodromy, easy to construct a homomorphism:

$$\begin{array}{c} \pi_1(\text{Period domain}) \longrightarrow \pi_0\, \mathrm{Symp}(X) \\ \downarrow \\ \mathrm{Aut}(\mathcal{F}(X)). \end{array} \qquad (*)$$

(ii) By contrast, very hard to build the map

$$\pi_1(\text{Period domain}) \to \mathrm{Aut}(D(X^\circ)),$$ (& they can show it's onto Aut_{CY}).

(However, we have no techniques in symp. topology to show is onto!)

contrast:

Note:
$$\begin{array}{ccc} \pi_1(\text{Period}) & \longrightarrow & \mathrm{Aut}(\mathcal{F}(X,\omega)) \\ \| & & \vdots \\ \pi_1(\text{Period}) & \xrightarrow[BB]{} & \mathrm{Aut}(D(X^\circ)) \end{array}$$

NOT obvious that this will commute.

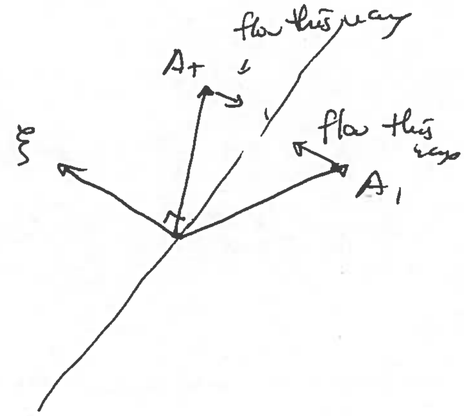
Rmk (2): Given semirigid $\mathcal{E}$; any stability condition σ has

$$\cancel{Z(\sigma)}\ Z_\sigma(K) = (\Omega, \mathrm{ch}(K))_{\mathrm{Muk}}, \quad \Omega \in N(X).$$

Conditions on Ω say $\mathrm{Re}(\Omega)$, $\mathrm{Im}(\Omega)$ span a pos.-def. 2-plane in $N(X)_{\mathbb{R}}$, so the 1-dim'l orthogonal is spanned by a vector Θ. Say in σ $\mathcal{E}$ has semistable "extremal" factors of phases $\phi_\pm$ (w.r.t. HN filtration). (rank 1 $\Rightarrow$ Pic rank 1)

Solve $\dfrac{d\Omega}{dt} = \xi\Theta$,

(if flow for all time exists, gives BB thm (**)?)



~~eneral facts~~ General facts

The generic quartic $/\Lambda$ ← Novikov ring has Picard rank 1, and we have in our HMS statement a family of quartics

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow & & \\ \mathbb{A}^{19}_{\Lambda} & \xrightarrow[\text{val}]{} & \mathbb{R}^{19} \\ \cup & & \end{array}$$

val ← takes Nov. coeffs. of monomial

loci of quartics of higher Picard rank (well known to be algebraic subvarieties) ⟶ images lie in a $\mathbb{Q}$-linear hyperplanes.

here ! ⇒

In HMS statement we can ensure that $X^\circ_{d(\omega)}$ has Picard rank 1 by choosing $[\omega]$ to be irrational.

The K3 $/\Lambda$ is defined over a f.g. extension $\mathbb{K}_0$ of $\mathbb{Q}$, & such fields embed into $\mathbb{C}$ (Lefschetz principle)

e.g. [by Huybrechts's book]: $\mathrm{Aut}_{eq}(X^\circ_{d(\omega)}) \underset{1:1}{\longleftrightarrow} \mathrm{Aut}_{eq}(X^\circ_{\mathbb{C}})$

Template proofs: Use B-B, being careful.

e.g., $L \subseteq (X,\omega)$ Lagn torus w/ Maslov ~~zero~~ $\mu_L = 0$; it has a mirror object $\mathcal{E}^\circ$, which is semirigid on $X^\circ_{d(\omega)}/\Lambda$.

↑ irr. K3

Semistability of $\mathcal{E}$ for some stability condition $\sigma \in \mathrm{Stab}$

$\Rightarrow ch(\mathcal{E}) \neq 0 \quad \Rightarrow \quad [L] \neq 0.$

But, $(ch(\mathcal{E}), ch(\mathcal{S})) = \pm 1$ for some spherical object $\mathcal{S}$,

(check using classical results of Mukai; use FM trans. to move this object to a point).

& one needs to understand on A-side why this suffices
(using special knowledge of lattice of $ch(\mathcal{S})$, spherical objects

~~Result~~

<u>Another story</u>: Recall that if $Y^0 \subseteq \mathbb{P}^5$ is a cubic 4-fold, then

$$D(Y^0) = \langle \mathcal{A}_{Y^0}, \text{exceptional objects} \rangle.$$

$\uparrow$ 2dCY, Kuznetsov.

Let E be (Fermat) elliptic curve $\{x^3+y^3+z^3=0\}$.

Then, $(E \times E)/\mathbb{Z}/3$

$\uparrow$ antidiagonal action; hence preserves hol. 2-form on $E \times E$
(action preserving hol. 2-form, hence $\Rightarrow$)

The resolution $\widetilde{(E\times E)}/\mathbb{Z}/3$ is a K3 surface.
Z"

(9 A_2 singularities; resolve by 18 exc. curves).

(this has Picard rank 20, 9 $\times$ A_2-resolutions)

<u>Expectation</u> (work in progress):

$$D^\pi(Z, \omega) \simeq \mathcal{A}_{Y^0, \alpha(\omega)}.$$

<u>Why?</u> $E \xrightarrow{/\mathbb{Z}_3} \mathbb{P}^1$ gives

, so $(E\times E)/\mathbb{Z}/3 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$
$\cup$ open containing $L \times L$.

& study coverings, compactifications, resolutions. Do more work to deal w/ immersed rather than embedded.

Possible interest:

(i) If $[\omega]$ is irrational (as before), then $\mathcal{A}_{Y^0_{d(\omega)}}$ can't be geometric "for lattice theory reasons" (c.f. work of Addington–Thomas, ...)

$\mathcal{A}_{Y^0}$ has no object which behaves like '$\mathcal{O}_{pt}$'.

(ii) On symplectic side, (Z,ω) contains no Lag'n torus of $\mu=0$ which is essential in homology (e.g., couldn't contain a special Lag'n torus).

(this is a case where usual SYZ / mirror family Floer story may struggle)

(iii): generic irrational ω: $\mathrm{Auteq}(\mathcal{A}_{Y^0}) \cong \mathbb{Z}/3$ up to shift $[2]$.

(as we've been doing in the talk all along when talking about Aut).