J. Zhao, Periodic SH and the Hodge Filtration. 11/9/2016

(I) HMS for open manifolds

A-side

M open sym. (exact, monotone)

e.g. $M = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

B-side

$W : M^\vee \xrightarrow{\text{holo.}} \mathbb{C}$

$\hookrightarrow$ cx. alg. var.

e.g. $W : \mathbb{C}^3 \xrightarrow{xyz} \mathbb{C}$,

open string $\mathcal{W}(M)$ $MF(W)$

closed string $HH_\bullet(\mathcal{W}(M)) \cong SH^{\bullet+n}(M)$ (Ganatra).

$HH_\bullet(MF(W))$

$\overset{?}{\parallel}$

$\mathbb{H}^\bullet(M^\vee, (\Omega^\bullet(M^\vee), \wedge dW))$!

& $HP^\bullet(MF(W))$

$\cong \mathbb{H}^\bullet(M^\vee, (\Omega^\bullet(M^\vee)((u)), u\,d_{DR} + \wedge dW))$.

[Efimov, $M^\vee$ quasi-proj., $\mathrm{Crit}(W) \subset W^{-1}(0)$].

In fact, if a DG category A is smooth and proper, then [Katzarkov-Kontsevich-Pantev] ~~put~~ define a non-commutative Hodge structure on $HP_\bullet(A)$,

Problem: $\mathcal{W}(M)$ is smooth but NOT necessarily proper. (i.e. $H^*(\hom_{\mathcal{W}}(L_0, L_1))$ may be ∞-rank)

(& mirror: If W does not have proper critical locus).

Goal: propose a new definition of $HP_\bullet(A)$ for $A = \mathcal{W}(M)$

(II) Cyclic homology of mixed complexes.

Def: A mixed complex is: (C, d, Δ)

- C is a $\mathbb{Z}/2$ graded $\mathbb{K}$-mod
- d, Δ odd degree differentials on C s.t. $d\Delta = -\Delta d$.

Ex: A: X finite dimensional mfold, w/ $S^1 \circlearrowright X$,

$\Rightarrow$ $C_*^{sing}(X)$ is a differential graded module over $C_*(S^1) \underset{dga}{\simeq} H_*(S^1)$ $= \mathbb{K}[\varepsilon]/\varepsilon^2$ $|\varepsilon| = 1$.

dga using $S^1 \times S^1 \to S^1$.

$\rightsquigarrow$ get operation

$$\Delta = \cdot \varepsilon : C_*(X) \to C_{*+1}(X)$$

w/ $\Delta^2 = 0$. & dg module condition

$\Rightarrow d\Delta + \Delta d = 0$.

Ex B: $W : M^\vee \to \mathbb{C}$, say $M^\vee$ affine, W has isolated singularity.

$\rightsquigarrow \left(\Omega^{\bullet}_{hol,\ deRham}(M^\vee),\ \underset{\Delta}{\underset{\shortparallel}{\wedge dW}},\ d_{dR} \right)$ is a mixed complex.

(Rmk: we're only assuming Δ, d "odd"; rather than degree ± 1, which is why (dW, & d_{dR}) ~~[illegible]~~ aren't both $+1$).

Def: Given a mixed complex (C, d, Δ)

$\rightsquigarrow$ $C^- := C[[u]]$ formal power series in u w/ coeffs. in C.

$C^\wedge := u^{-1}C[[u]] = C((u))$ formal Laurent series

$C^+ := C((u)) / uC[[u]]$

then, on each complex, define: $\partial^{S^1} = d + u\Delta$, $(d^{S^1})^2 = 0$.. by assumption; hence. gives a differential on each cplx.

So, now can take:

$H_*(C^-, d^{S^1})$ negative cyclic homology

$H_*(C^\wedge, d^{S^1})$ periodic cyclic homology

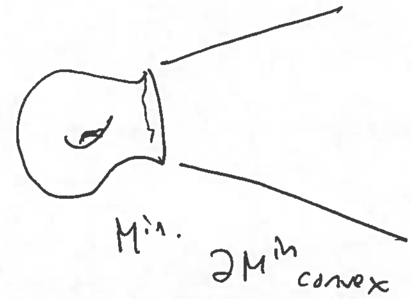
$H_*(C^+, d^{S^1})$ (positive) cyclic homology.

(III) Periodic symplectic cohomology.

- Consider an exact symplectic manifold (M,θ), which is conical at ∞.

so $M = \underset{d\theta = \omega \text{ sympl.}}{M^{in}} \cup_{\partial M^{in}} [1,+\infty)_r \times \partial M^{in}$, with $\underbrace{[1,+\infty)_r \times \partial M^{in}}_{d(r\alpha)}$

∂M^{in} is a convex hypersurface, e.g. $\theta|_{\partial M^{in}} = \alpha$



Ex: $M = T^*Q$, Q closed Riemannian,

$$\theta = \sum p_i dq_i, \quad M^{in} = D^*_{\leq 1} Q.$$

Given a Hamiltonian $H_t : M \to \mathbb{R}$, can look at

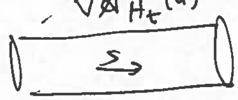
$\mathcal{A}_{H_t} : \mathcal{L}M \to \mathbb{R}$ is a Morse function.

$CF^*(H_t) = \mathbb{Z}\langle \text{crit. pts of } \mathcal{A}_{H_t} \rangle = \mathbb{Z}\langle \text{Ham. orbits of } X_H.$

↑ sympl. co-chain cplx,

The differential counts

$\nabla \mathcal{A}_{H_t}(u) = -\partial_s(u).$



x^- x^+

$d : CF^* \to CF^{*+1}.$

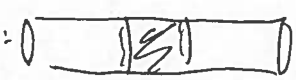
$\exists$ an S^1-action on $\mathcal{L}M$ $\theta \cdot x(t) = x(t+\theta)$

$\Rightarrow$ the BV operator counts solutions to

$\Rightarrow$ induces

$\underline{\underline{\Delta}} : CF^*(H_t) \to CF^{*-1}(H_t)$

$\{ (\theta, u) \mid u :$



$\to M$

H_t $\quad H^\theta_{s,t}$ $\quad H_{t-\theta}$

$S^1 \circlearrowright (\theta,u) \mapsto \theta \in S^1 \ni \theta$

solves $\nabla \mathcal{A}_{H^\theta_{s,t}}(u) = \partial_s u. \}$

$\Rightarrow$ have

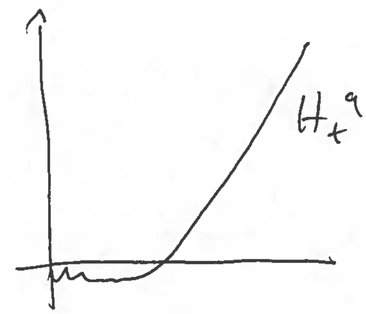
$$d^{S^1} = d + u\Delta \; (+ u^2 \Delta^{(2)} + \cdots)$$

b/c $\Delta^2 \neq 0$ on chain level generally, but ignore this.)

Seidel [Bourgeois–Oancea].

Case 1: the usual (completed) periodic symplectic cohomology.

$$\widehat{HP}(M) := H_*\left(CF^*(H_t^q)((u)),\ d^{S^1}\right).$$



Problem: is usually infinitely generated; since we're completing here, might run into completion problems;

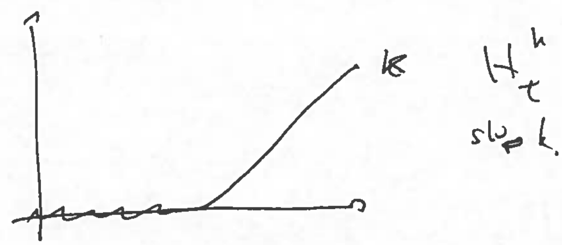
Ex: when $M = T^*Q$, Goodwillie's theorem

$\Rightarrow$ $\widehat{HP}(M)$ only depends on $\pi_1(Q)$.

So e.g. if Q is simply connected, then $\widehat{HP}(M) =$ rank 1.

Case 2: Take

$$\varinjlim_k H_*\left(CF^*(H_k)((u)),\ d^{S^1}\right) =: HP^*_{loc}(M)$$

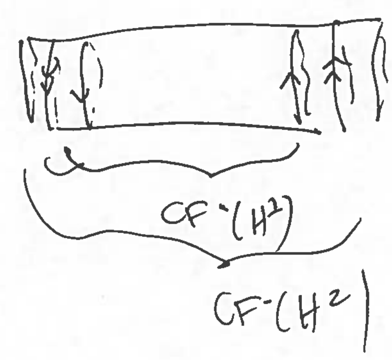


e.g. $M = T^*S^1$

Thm: [Z., 2014]: Given an exact symplectic manifold, then

$$HP^\bullet_{loc}(M) \cong H^\bullet(M)((u))$$

using $\mathbb{Q}$-coefficients.



Rmk: If M is not exact (e.g., monotone), (but say exact near ∞) then a similar argument should show

$$HP^*_{loc}(M) \cong QH^*(M, \Lambda_K)((u))$$

when $\mathrm{char}(K) = 0$. (iso. given by usual PSS map).

(IV) The Hodge filtrations.

~~FP~~ Given H_k,

$\cancel{F^p HP^{even}_{loc}(M) = \{[a]\}}$

$\uparrow$

~~induced fr~~ ~~fathe~~ $\cancel{H^*(F^p CP^{even}_{loc}(M))}$

$F^p CP^{even}_{loc}(M) =$ limit of

$F^p CF^{even}(H_k)((u)) = \{\sum_{i \geq p} a_i u^i \mid a_i \in CF^{even}(H_k)\}$

$\}$ direct limit over all H_k.

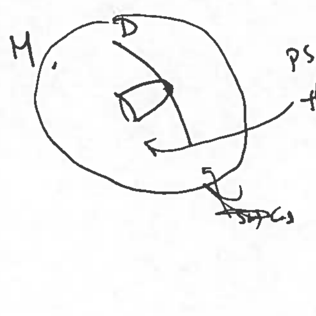
descends to filtration on $HP^{even}_{loc}(M) \underset{\mathbb{Q}}{\cong} H^*(M)$.

These are interesting & non-trivial filtrations;

- If $M = X \setminus D$, X monotone, $D = c_1(X)$,

[Seidel]: The Hodge filtration $HP^\bullet_{loc}(M)$ encodes certain GW invts, (possibly ~~potentially~~ w/ gravitational descendants coming from D).

An easy example: (look at $\mathbb{C}$):



PSS: this class lifts to equivariant $SH^0_{S^1}$, lies in $F^{-1}\widehat{SH}^*_{loc}$ it seems? & look at filtration piece this class lies in; such fact (of this class) seems related to GW invts.