11/10/2016

Z. Sylvan, Partially wrapped Floer theory

Idea: Enhance wrapped Floer theory using data on ∂X, X Liouville domain

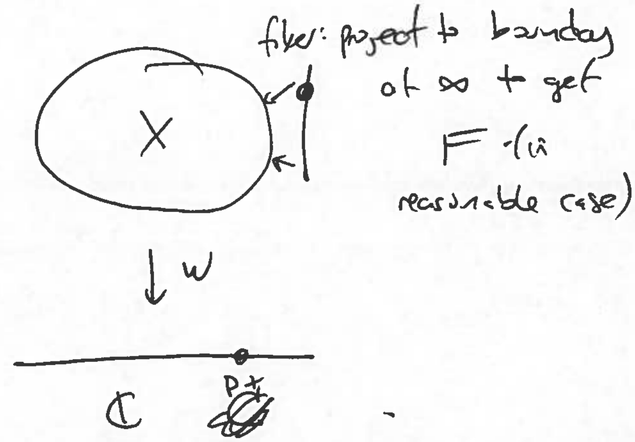
($\xleftrightarrow[\text{mirror}]{}$ "(partial) compactification data for $X^\vee$?").

Def: A stop $\sigma \subset \partial X$ is a hypersurface w/ ∂ s.t.

$(\sigma, \lambda_X|_\sigma)$ is a Liouville domain.

E.g.: • page of an open book on ∂X.

• $W^{-1}(pt)$ for $W: X \to \mathbb{C}$.
{near ∞}



• Smooth legendrian $L \subset \partial M$, thickened in contact direction. (modeled on $T^*L \subset J^1L$)

• Expected: same with singular L in particular for

$\partial \Lambda$ ← Lag. skeleton of X.

Idea for Floer theory w/ σ

Pick Reeb vector field adapted to σ i.e., • σ still symplectic,

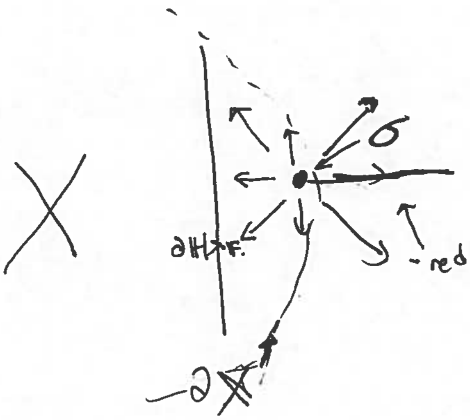
• R parallel to $\partial\sigma$

$\Rightarrow$ all trajectories ~~thru~~ either in $\partial\sigma$,

or have some intersection number $n_\sigma \geq 0$.

"partially wrapped" just means $n_\sigma = 0$
(& no trajectories in $\partial\sigma$).

More accurately: use local model $\hat{F} \times \mathbb{H} \subset \hat{X}$ ← completion.
↑ large half space



$\hat{F}_\sigma \times \mathbb{H} \subset \hat{X}$.

proper codim. 0 embedding

In this local model, have a central fiber F_0;

want to choose H & J "fixing F_0"

($\rightsquigarrow$ positivity of intersections w/ F_0).

Open string case: get a filtration (only considering Leg's missing σ).

$$CW^*_\sigma := CW^*_0 \ (\subset CW^*_1 \subset \cdots \subset CW^*).$$

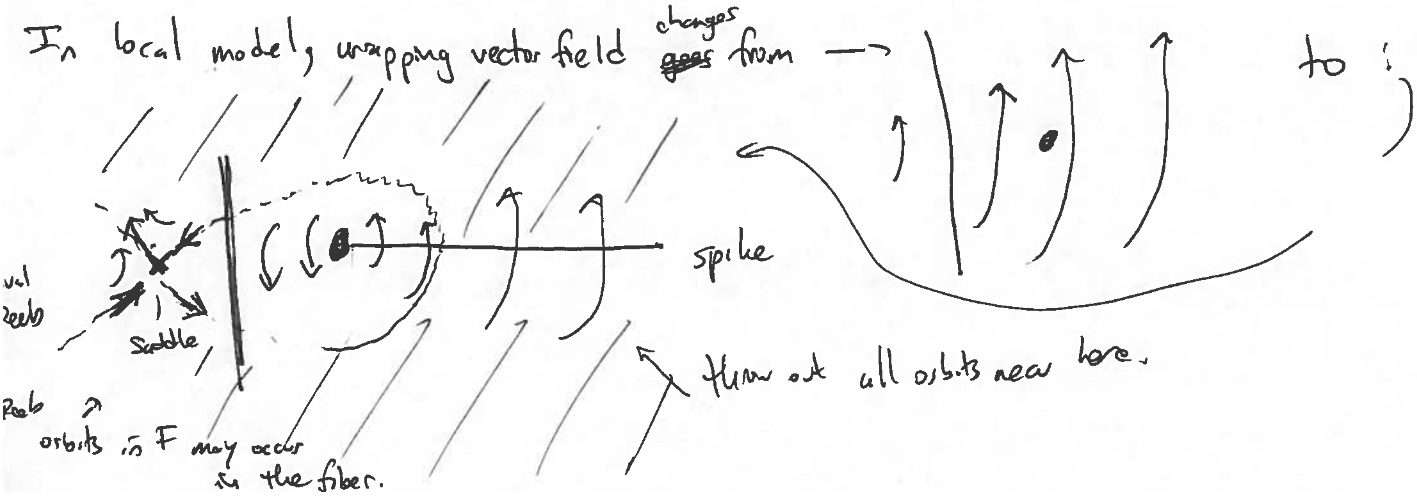
Closed string case: more complicated.

Def: Partially wrapped Fukaya category: $\mathcal{W}_\sigma(X)$ is an A_∞ category

$$\text{Ob} = \{ \text{ob of } \mathcal{W} \text{ which avoid } \sigma,$$
$$\text{e.g., } L \text{ exact w/ Leg. } \partial \text{ avoiding } \sigma \}$$

$$\hom^*(L_0, L_1) = CW^*_\sigma(L_0, L_1) \longleftarrow \text{morphisms avoiding } \sigma.$$

In local model, wrapping vector field ~~goes~~ changes from $\longrightarrow$



Expectations: (all up to Tw^π)

- $\mathcal{W}_{\sigma_W}(X) \cong FS(W)$

  ↑ stop assoc. to superpotential

  (point: thimbles don't see funny dynamics at ∞ on fiber F of saddle $\times$, b/c they're proper on F).

- $\mathcal{W}_{\sigma_L}(X) \cong LDGA\langle L, C_*(\Omega L)\rangle$

  ↑ non-central coeffs; so coeffs. don't commute

  (if $\mathcal{W}(X) \cong 0$) [cf. Ng's situation].

  (b/c in this case $\mathcal{W}_{\sigma_L}$ gen. by one object (for each component of L), whose endomorphisms match)

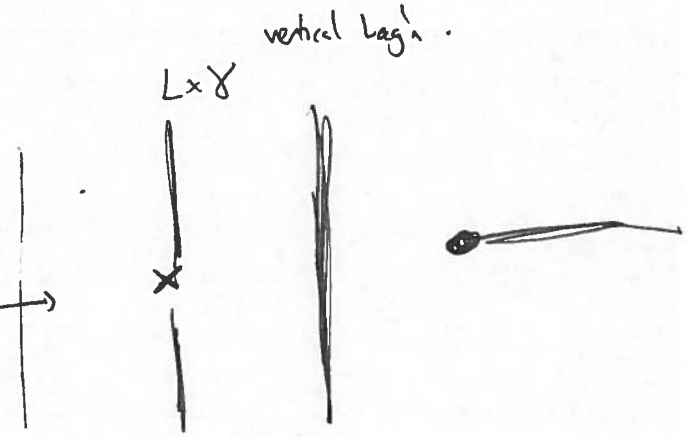
- $\mathcal{W}_{\sigma_\Lambda}(X) \cong \mu Sh^w_\Lambda(X)$ (Nadler.)

  ↑ skeleton for X now, and the stop, $\sigma_\Lambda = N_\varepsilon(\partial^\infty \Lambda)$)..

- $\exists$ inclusion functor

  $\iota_\sigma : \mathcal{W}(F_\sigma) \longrightarrow \mathcal{W}_\sigma(X)$ via: $L \mapsto$

  (Orlov functor).



Thm: (S.): If F_σ is "strongly non-degenerate," then the map

$\mathrm{coker}(\iota_\sigma) \longrightarrow \mathcal{W}(X)$ is fully faithful.

(note: everything in image of ι_σ is a zero object in $\mathcal{W}(X)$, b/c displaceable).

Eg.: The following F_σ are strongly non deg:

- Open surfaces (other than D^2)
- Cotangent bundle (Abouzaid)
- Expect: Weinstein domains (admitting nice skeleta) [Ganatra-Pardon-Shende].

Ex: if $\mathcal{W}(X)=0$, $\Delta \subseteq \partial X$ smooth,

take $\iota_\sigma(T^*_q\Delta) =: L$, expect $\mathrm{End}_{\mathcal{W}_\sigma}(L,L) \cong$ 2DGA $\langle L, C_*(\Omega L)\rangle$.

Stronger thm in progress: (S, GPS):

Say have (M_1,σ_1), (M_2,σ_2) have an isomorphism of fibers

$F_{\sigma_1} \cong F_{\sigma_2} = F$; and everything is strongly nondegenerate.

Then, can form $M = M_1 \#_{\sigma_1 \sigma_2} M_2$. Get maps

$$\begin{tikzcd} & \mathcal{W}_{\sigma_1}(M_1) \arrow[dr] & \\ \mathcal{W}(F) \arrow[ur, "\iota_{\sigma_1}"] \arrow[dr, "\iota_{\sigma_2}"'] & & \mathcal{W}(M) \\ & \mathcal{W}_{\sigma_2}(M_2) \arrow[ur] & \end{tikzcd}$$

is a pushout square.

If M is not strongly degen, get fully faithful map

pushout $\to \mathcal{W}(M)$.

Example: Viterbo functoriality

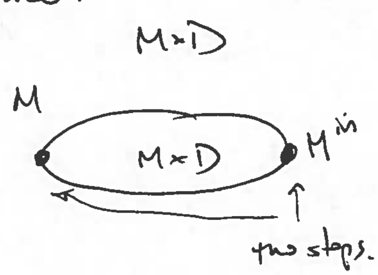
Given a codim 0 embedding of LD $M^{in} \subset M$, get "Viterbo transfer maps"

$SH^*(M) \longrightarrow SH^*(M^{in})$ (Viterbo)

(Kragh)

$\mathcal{W}(M) \longrightarrow \mathcal{W}(M^{in})$ (Abouzaid-Seidel),

Consider:



$\rightsquigarrow \iota_M, \iota_{M^{in}}$

Result of computation: $\iota_{M^{in}}: \mathcal{W}(M^{in}) \to \mathcal{W}_\sigma(M \times D)$ is fully faithful. with no assumptions.

Cor: If M^{in} is strongly non-degen, then

$Tw^{\pi}(\mathcal{L}_{M^{in}})$ is a quasi-equiv.

$\mathcal{L}_{M^{in}}$

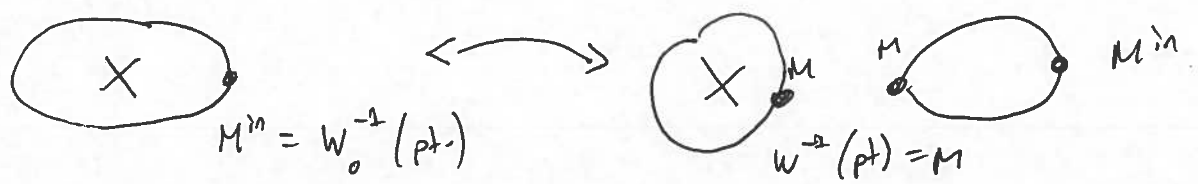
$\Rightarrow$ Can recover $V_{\Phi} = \left(Tw^{\pi}(\mathcal{L}_{M^{in}})\right)^{-1} \circ \mathcal{L}_M$.

Application: If W_0 is a subpolynomial of a superpotential W (= sum (monomial))

(some of the monomials)

$D \supset \bigcup_{\text{smaller}} D_\alpha$, $\qquad D = \bigcup D_\alpha$

then there is a Liouville embedding $W_0^{-1}(pt.) \hookrightarrow W^{-1}(pt)$; and can decompose



$\rightsquigarrow$ map from $\mathcal{W}_{\mathfrak{S}_W}(X) \longrightarrow \mathcal{W}_{\mathfrak{S}_{W_0}}(X)$.

(maybe seeable in terms of rescaling W_0 rel. other monomials, & seeing a semi-orthogonal decomposition)