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M compact symplectic

$\rightsquigarrow$ dg category w/o using hol. disks (defined only using sheaf theory)

via

$T^*X \rightsquigarrow sh(X) \subset shv(X\times\mathbb{R})$

$\{F \in shv(X\times\mathbb{R});\ \Gamma_c(U\times(-\infty,a), F) \sim 0\}$ ($U \subseteq X$ any open)

$\forall\ U$ open., $\forall a$.

Can interpret an object in $sh(X)$ as a filtered sheaf on X w/ some condition.

Assumptions

1) M is compact.

$[\omega]$ has integral periods $\Rightarrow \exists$ $\begin{matrix} S^1 \subset L \\ \downarrow \pi \\ M \end{matrix}$ ~~one which~~ w/ $[\omega] = c_1(L)$.

e.g. exists $\theta \in \Omega^1(L)$ S^1-equivariant, &

$$\begin{cases} \theta\big|_{\frac{\partial}{\partial\varphi}} = \cancel{2}\,1, \ \& \\ d\theta = \pi^*\omega. \end{cases}$$

θ turns L into a contact manifold

& look at ~~Cont S^1-equiv (L)~~ $\underset{\shortparallel}{G}$ inf. dim'l Lie group
$\mathrm{Ham}(M) := \{F \in \mathrm{Diffeo}(L);\ F$ is S^1-equiv, $\& F^*\theta = \theta\}$.

(up to perturbation, any elt of this group is always caused by a Ham. vector)

Have a map $\mathfrak{g}^*$ ~~$\leftarrow M$~~ $M \subset \mathfrak{g}^*$ = "functions on M"

Can look at T^*G $\quad$ $\bar{M}\times M$ $\quad$ $T^*G = G\times\mathfrak{g}^*$

$$\begin{array}{ccc} T^*G & & \bar{M}\times M \\ & \nwarrow F \quad \nearrow^{m \ \ gm}_{(id,\ G\text{ action})} & \\ & L & \end{array}$$

(g,m) Lagn corresp.

& $F(L) = G\times M$

~~If Betty~~ then, on the nose, $M^-\times M$ gets realized as a Ham. reduction of coisotropic in ~~X~~ T^*G

Look at

$$sh(G) \supset sh(G)_{[F(L)]}$$

$\uparrow$ singular support $= G \times Cone(-)$.

$$\begin{array}{c} L \\ \downarrow \\ M \times M \\ x \quad y \end{array}$$

has fiber $L_{(x,y)} = \{ g \in G \mid gx = y \}$.

doesn't answer.

$$\begin{tikzcd} \Omega M \arrow[d] & \\ E \arrow[r, two heads] \arrow[d, bend right, "\text{replace contractible}"'] & G \\ x \times M & \end{tikzcd}$$

<u>Un doubling</u>: Fix $B_R \subset M$:

$$\begin{tikzcd} T^*G & & M \times M \supseteq B_R \times M \\ & L \arrow[ul, "i"] \arrow[ur, "p"'] & \end{tikzcd}$$

$$sh(G.)\left[i p^{-1}(B_R \times M) \right]$$

want to undo this, b/c every sheaf on B_R is <u>torsion</u>; has finite capacity $\mathcal{C}$ is killed by it:

$${}_{B_R} sh(G)_M$$

Assume $B_R \subset B_{R'}$

Have a group $G_0 \subset G = g$: supp $g \Subset B_R$!

$\uparrow$ cptly supported symp.

Can do now

$${}_{B_R} sh(G_0)_{B_R} \cong sh(\mathbb{R}^n \times \mathbb{R}^n)_{[B_R \times B_R]}$$

and also

acts on $sh(\mathbb{R}^n)_{[\text{int } B_R]}$

ach

$$ {}_{B_R}sh(G)_{B_R} \curvearrowleft {}_{B_R}sh(G)_M $$

& have $\quad {}_{B_R}sh(G)_{B_R} \longrightarrow {}_{B_R}sh(G_0)_{B_R} \circlearrowleft sh(B_R)$.

$\mathcal{M} \circlearrowleft sh(B_R) \otimes_{{}_{B_R}sh(G)_{B_R}} {}_{B_R}sh(6)$ ~~M~~ constant objects are ~~[illegible]~~ objects, so can use this category for proving nondisplaceability

find sth. acting nec. to undo

have G equiv, so stable under G action by right.

(actually, replace ~~G~~ G by E = 'space of based paths' in G),

& want to "undo" ~~E~~ action.

can describe to some extent. $\quad \mathcal{M} := \mathrm{End}_{{}_{B_R}sh(G)_{B_R}}(sh(B_R))$

where $\quad \mathcal{M}_{cl} \cong Loc(\Omega M)_{tw}$? twist / what twist? [Lurie, Jin–Treumann]

group-y convolution

(nothing interesting over 2-per. complexes)

loc. systems are global sections of trivial sheaf on M,

& need to take locally free sheaf on ΩM, & ... like shifts along Novikov axis;

sheaf of spectra & compatible w/ G action;

really $Pic(S) \times \mathbb{R}$

classifying data of sheaf is a map

Novikov ring of spectra

$B^2 Pic(S)$.

spectra or actually

$$M \longrightarrow B^2 Pic(\cancel{S}).$$

this only depends on

$M \longrightarrow BSp(2n)$

J-hom., Bott, etc.

deloop

gives, $\Omega M \longrightarrow B\,Pic\,S^0$ group map: gives a monoidal category.

Need to find $\mathcal{M}$

Guess what $\mathcal{M}_{cl}$ is, & solve deformation problem to lift to

$$\mathcal{M} := \mathrm{End}_{\cdot_{BBs}\,R^{sh(G)}_{B_R}}(sh(B_R))$$

what's $\mathcal{M}_{cl}$? have alg in monoidal category

$\mathcal{M}_{cl}$? $\mathcal{M}_{cl} \; Loc(\Omega M)_{tw}$ ← const. sheaf

& take $\mathcal{M}_{cl}$-Mod (modules over alg in monoidal cat)

Now, lift to quantum level.

Ingredients:

$\mathcal{M}$ monoidal cat. (or more generally $(\infty,2)$ cat). "bisimplicial set"

Then, $\int_{S^1} \mathcal{M}$ is $(\infty,1)$.

"bisimplicial set" $\xrightarrow{\text{simplicial nerve}}$ simplicial space.

decategorifies

$(\infty,1)$ decat.

$\mathcal{M}$ usual alg cat. $\to$ $C_\cdot(\mathcal{M})$ $(\infty,0)$ category.

↑ Hochschild chains of category

$F \in \mathcal{M}$ mod. dualizable

functor (by invertible 2-morphisms)

$ch(F) \in \int_{S_1} \mathcal{M}$ object of cat.

Now, if one has a dualizable module over $\mathcal{M}$,

$$F \longrightarrow ch(F) \in C_\cdot(\mathcal{M})$$

$(\infty,2)$: Get a functor $\mathcal{M}^\times \xrightarrow{ch} \left(\int_{S_1} \mathcal{M}\right)^{S^1}[u^{-1}]$

invert 2-morphisms (underlying $(\infty,1)$ category)

"periodic cyclic homology".

(downstairs, has an analogy; cyclic homology.

$\downarrow$ u loc. by

$\left(\int_{S^1} \mathcal{M}\right)^{S^1}$ nice advantages

Point: $\cdot\left(\int_{S^1} \mathcal{M}\right)[u^{-1}]$ satisfies ~~crystalline property~~ / homotopy invariance ;

so objects in ~~M~~ $\mathcal{M}_{cl}$ are same as in ~~M~~ $\mathcal{M}$.

$$\mathcal{M}^\times \xrightarrow{ch} \left(\int_{S^1} \mathcal{M}\right)^{S^1}[u^{-1}]$$

$$\cong$$

$$ch(\mathcal{U}_{cl}) \in \left(\int_{S^1} \mathcal{M}_{cl}\right)^{S^1}[u^{-1}]$$

$$\in$$

TR

↑ what are objects here? they're traces on M; maps $\mathcal{U} \to$ Vect invariant / in cyclic homology.

↑ this map exists for free.

why does one need it? want to encode fact that $QH^\cdot$ has trivial TR.

If define TR property, ~~$QH^\cdot(X)_{cl} = \hom$~~

1) $\hom_{cl}(TR, ch(\mathcal{U}_{cl})^\cdot) \cong C^*(M)((u))$ w/ trivial circle action.

2) $End(TR) \otimes_{k[u^{\pm 1}]} \mathbb{Q} \sim 0$.

(so, can't get two different traces by this

have a chosen fundamental class

$TR \xrightarrow{\text{quantum}} \mathcal{U}$; & this quantizes by crystalline property

Want: Find $\nu \in$ ~~M~~ $\mathcal{M}$-mod

& $ch(\nu) \xrightarrow{\sim} ch(\mathcal{U})$.

$\uparrow$ TR

Why does this problem now have a solution?

Say have

$$f: X \longrightarrow Y \text{ over } \Lambda$$
$$\quad\quad\quad\quad\quad \Psi\ P$$

$$X_0 \longrightarrow Y_0 \supset \quad \& \ F(p_0) \xrightarrow{\sim} p_0.$$

say, have classical retraction.

q_0 ← classical preimage

$$\Rightarrow \mathrm{End}_{X_0}(q_0) \longrightarrow \mathrm{End}_{Y_0}(p_0)$$

$\Rightarrow$ Then, there is a canonical way to lift

$$X_0 \rightsquigarrow X.$$

Then, define category in right way.

X will be all triples

$$\begin{tikzcd} ch(v) \arrow[r] \arrow[d, "forget"'] & ch(u) \\ & TR \arrow[u] \\ ch(u) & \end{tikzcd}$$

& Y will be

$$\begin{tikzcd} ch(u) \\ TR \arrow[u] \end{tikzcd}$$

Calculate endomorphisms, it will work?