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Family Floer homology & mirror symmetry

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(X, ω) closed sympl m'fld

$\downarrow$ Lagr. torus fibration, (no singular fibers).

Ex: ($\mathbb{R}^{2n}$/Nilpotent "Heisenberg group")

Q Assume $\pi_2(Q) = 0$ (~~this~~ this seems to be the case in all examples?)

$\Rightarrow \exists$ an analytic space Y (mirror) with a class $\beta \in H^2_{an}(Y, \mathcal{O}^*)$
s.t. we have a fully faithful embedding

$$\mathbb{F}(X) \hookrightarrow D^b_\beta(\mathrm{Coh}\, Y)$$

Construction of Y: due to Fukaya, ~~...~~
+ local construction

'14: A. ~~proved~~ constructed A_∞ functor + proof ~~that~~ of faithfulness

Today: "fully" (unfortunately, much harder).

Remarks:

- In general, must work in analytic category.
  e.g. (Thurston, A - Auroux - Katzarkov) $\exists$ a ~~of a~~ sympl 4-m'fold w/ $b_1 = 3$ (Thurston: non-Kähler)
  w/ a Lagn torus fibration $\Rightarrow$ Mirror is an analytic space
  which is $E \to Y$, $\downarrow$ ell. fibration, E ell curve, w/ no section (Kodaira, who proved these are non-projective (algebraizable)).
- We have to study analytic spaces over $\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = +\infty \right\}$
  $\mathbb{K}$ = base field (can do this b/c the arbitrary $\pi_2(Q) = 0$)

Have: $\mathrm{val}: \Lambda^* \longrightarrow \mathbb{R}$
" $\Lambda \setminus \{0\}$

$\sum a_i T^{\lambda_i} \longrightarrow \min_{a_i \neq 0} \lambda_i$

$a_0 T^{\lambda_0} +$ higher order $\longrightarrow \lambda_0$

analogy:
$\Lambda \sim \mathbb{C}$
$\cup$
$z \in \mathbb{C}^* = e^{\rho + ib}$
$\downarrow \quad \downarrow$
$\log|z| \quad \mathbb{R}$
ex. to norm

We will consider a special kind of domain $\subset [\Lambda^*]^n$, defined by inequalities on the valuation, such as

$$\big\downarrow \text{val}$$

$$P \subset \mathbb{R}^n.$$

Laurent monomial

$\alpha \in \mathbb{Z}^n$ $\boxed{\text{val } z^\alpha \geq \lambda}$, to obtain closed compact polytope $P \subset \mathbb{R}^n$.

We could also think about it backwards:

$\mathbb{R}^n$ is equipped w/ the standard lattice $\mathbb{Z}^n$. Define an integral affine polytope to be a polytope $P \subset \mathbb{R}^n$ defined by equations of the form $\langle u, \alpha \rangle \geq \lambda$, $u \in \mathbb{Z}^n$, $\lambda \in \mathbb{R}$.

The ring Γ_P of analytic functions on Y_P consists of all

Laurent series $\left\{ f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \;\middle|\; \begin{array}{l} c_\alpha \in \Lambda. \\ f \text{ converges in } Y_P \end{array} \right\}$

$$\Updownarrow$$

$$\lim_{|\alpha| \to \infty} \text{val } c_\alpha y^\alpha = +\infty \quad \text{for all } y \in Y_P.$$

Example: ⓪ $n = 1$, $P = \{0\}$

$y \in Y_P \iff \text{val } y = 0.$

$\Rightarrow \text{val } c_\alpha y^\alpha = \text{val } c_\alpha \Rightarrow \Gamma_0 \longleftrightarrow \text{val } c_\alpha \to \infty.$

① $P = [-1, 1]$.

get $\lim_{|\alpha| \to \infty} \text{val } c_\alpha - |\alpha| \longrightarrow +\infty$. (valuations go sufficiently fast to ∞.

Tate, – — developed a theory of rigid analytic spaces with such local models.

Q: where do such rings appear in symplectic topology?

Say M symplectic, $\underline{c_1} = 0$ (Assume)
$\cup$
L Lagr. $\underline{\text{Maslov}} = 0$

L together w/ a choice of unitary rank 1 local system $\rightsquigarrow$ object of $\mathcal{F}(X)$ (may be obstructed)

Rank 1 local system is a rep. of $\pi_1(L)$; i.e. module over $\Lambda[\pi_1 L]$; ~~being unitary~~ If $L = T^n$, this is exactly Laurent polynomials

$$\Lambda[\pi_1 L] \cong \Lambda[H_1(L, \mathbb{Z})] = \text{Laurent polys.}$$

A unitary rep. corresponds to a rank 1 - module over the completion of $\Lambda[\pi_1 L]$ obtained by $\sum c_g g$, $\text{val}\, c_g \to +\infty$. For torus, get Γ_0 from earlier

Thm (will eventually appear):

$\exists$ an enlargement of $\mathcal{F}(X) \subset \hat{\mathcal{F}}(X)$ ← objects are pairs (L, modules over completion of $H_*(\Omega L)$).

Go back to

$$\begin{array}{c} X \supset X_q \\ \downarrow \\ Q \ni q \end{array}$$

Arnol'd - Liouville $\Rightarrow$ X_q is a torus.

$\xrightarrow{\ }$ completion $\Lambda[H_1(X_q, \mathbb{Z})] \rightsquigarrow Y_q$ rigid analytic space of unitary rank 1 on X_q;

and define $Y = \bigcup_{q \in Q} Y_q$.

infinite union.

Key fact: (Fukaya): If one fixes a Lag'n L, and an a.c. structure J, then we can define a Floer homology group

$$HF^*(L, (X_q, \Gamma^P))$$

where P is a subset of $H^1(X_q; \mathbb{R})$, whenever the diameter of P $< \varepsilon$ ← depends on L, J.
P^c $\quad$ 0^L

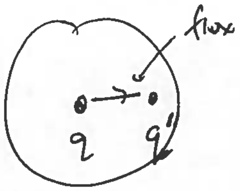
So, we want to define some category

$$\mathcal{A}(Q) \quad \text{(analytic)}$$

whose objects are $P \subset Q$ integral affine.

(~~Bundle~~ flux gives a local identification of Q with $H^2(X_q, \mathbb{R})$ near $q \in Q$.),

$\exists$ lattice $H^2(X_q, \mathbb{Z}) \subset H^2(X_q, \mathbb{R})$



such that

$$(*) \quad \mathcal{A}(P, P') \cong \Gamma^{P'} \text{ if } P' \subset P.$$

$\mathcal{A}(P, P')$ ← morphisms

Given L, assign a module over $\mathcal{A}$, call it $\mathcal{L}$.

(not directly always possible, but trick:)

Tate proved that Čech complexes of rings of functions are acyclic $\Rightarrow$

$\mathcal{A}_\varepsilon$ generates $\mathcal{A}$. So, just need to define a module of L over $\mathcal{A}_\varepsilon$;

$\mathcal{A}_\varepsilon$: polytopes of diam $< \varepsilon$.

by making above construction compatible with operations.

[Why do we want to assume $(*)$? ans!)

Having done this, we get, for L.

① A complex of Γ^P modules for each $P \subset Q$.

② If $P' \subset P$, ~~should have~~

$$\underbrace{\mathcal{A}(P, P')}_{= \Gamma^{P'}} \otimes \mathcal{L}(P) \longrightarrow \mathcal{L}(P').$$

induces a map

$$\Gamma^{P'} \otimes_{\Gamma^P} \mathcal{L}(P) \longrightarrow \mathcal{L}(P');$$ show this is an equivalence.

mirror side

Y_p 

$\Rightarrow$ complex on Y_p, when restricted to Y_p', is an equivalence

This is the data of a complex of coherent sheaves, except didn't check higher coherence. (triple intersections, etc.; this is where $H^2(Y, \mathcal{O}^*)$ appears)

On triple intersections, get potential in<u>compat</u>ibility

its easy to see $HF^*(L,L)$ acts on $HF^*(L, (X_q, \Gamma^p))$

Induces a map

$$HF^*(L,L) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{L}),$$

known to be faithful

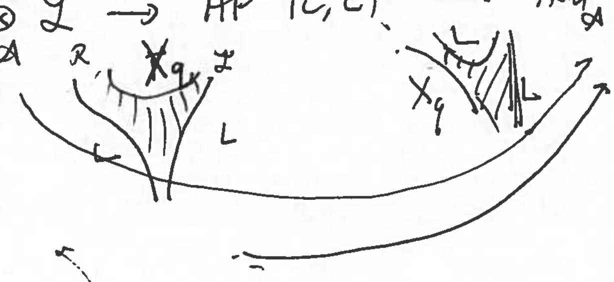
in principle, should write $\mathcal{A}_\varepsilon$ everywhere

To show <u>full</u>:

consider the right module $\mathcal{R} \rightsquigarrow HF^*((X_q, \Gamma^p), L)$

Have

$$\begin{array}{l} \mathcal{R} \otimes_{\mathcal{A}} \mathcal{L} \longrightarrow HF^*(L,L) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \\ \downarrow \end{array}$$



analyze:

(prove this) is an isomorphism or show key, prove its surjective

(what haven't we used yet?)

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \Delta) \otimes_{\mathcal{A}} \mathcal{L} \arrow[r] \arrow[dr] & \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \\ & \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \Delta \otimes_{\mathcal{A}} \mathcal{L}) \arrow[u] \end{tikzcd}$$

It seems the main thing to do is:

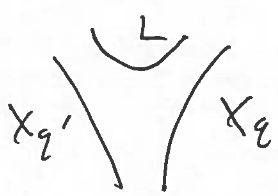
$$\begin{tikzcd}
R \otimes_{\mathcal{A}} \mathcal{L} \arrow[r] \arrow[d] & HF^\cdot(L,L) \arrow[d] \\
\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \Delta) \otimes_{\mathcal{A}} \mathcal{L} \arrow[r, "\text{always an isomorphism}"'] & \mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{L})
\end{tikzcd}$$

want → this is an isomorphism

hardest map

So, we want a map $\mathcal{L} \otimes_{\Delta} \mathcal{R} \to \Delta$.

this is:



If $q = q'$, this is easy.

If $q \neq q'$, then this should be zero. (problem! b/c these are identified.)

But I implicitly have been identifying

$$(X_q, \Gamma^P) \sim (X_{q'}, \Gamma^P) \quad \text{if} \quad P \subset \mathbb{Q}, \; \& \; q, q' \in P.$$

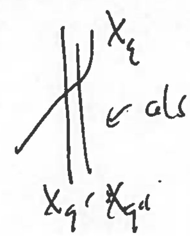
To resolve, we have to compute more morphisms in $\mathcal{A}$ (can't choose cplxes, has very hard lemma (using Banach topology on Γ^P) b/tw local systems in analytic geometry instead identify:) & get:

$$HF^\cdot((X_q, \Gamma^P), (X_q, \Gamma^{P'})) = 0 \text{ if } P \neq P'.$$

(this tells you some consistency checks are satisfied)

in this case, "point: ∃ morphisms but they're not continuous morphisms in Banach topology, & taking 'continuous homs' ??

In general, to make sense of ~~that diagram~~ the bottom left corner, perturb diagonal by a Hamiltonian diffeomorphism; becomes less dire.



← also true for nearby fibers.

Rmk: one remaining step