K. Fukaya 11/11/2016

$X \circlearrowleft G$ Assume: 0 is a regular value, & $\mu^{-1}(0) \circlearrowleft G$ is free.

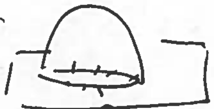
$\mu : X \longrightarrow \mathfrak{g}^*$ moment map

Then, $Y = \mu^{-1}(0)/G = X /\!\!/ G$.

Consider $L \subset \mu^{-1}(0) \subset X$ Lagr. submanifold in X,

& assume L is G-equivariant.

Then, $\bar{L} = L/G \subset Y$ is a Lag'n submanifold.

Consider $\beta \in H_2(X, L; \mathbb{Z})$, & $\mathcal{M}_{k+1}(\beta) = \left\{ \right.$  $\left. \right\}$ $(D^2, \partial D) \xrightarrow{u} (X, L)$ holo., stable

k+1 marked points on ∂.

Have:
$$\begin{tikzcd} & \mathcal{M}_{k+1}(\beta) \arrow[dl, "ev^k"'] \arrow[dr, "ev"] & \\ L^k & & L \end{tikzcd}$$

Since L acted on freely, $\mathcal{M}_{k+1}(\beta)$ has a free G-action.

& can consider:

Assume: has a Kuranishi structure.

$$\begin{tikzcd} & \mathcal{M}_{k+1}(\beta)/G \arrow[dl] \arrow[dr] & \\ (L/G)^k = \bar{L}^k & & L/G = \bar{L} \end{tikzcd}$$

(in general, want to use Borel construction, & remember action of $H^*(BG)$ on "$H^*(\bar{L})$" $= H^*_G(L)$

(even if $L \circlearrowleft G$ free? get map $\bar{L} \to BG$? $\bar{L} \to M \to BG$ (can ??)).

Then, get:
$$m_k^G : \Omega(\bar{L})^{\otimes k} \longrightarrow \Omega(\bar{L}) \otimes \Lambda. \quad \text{satisfying } A_\infty \text{ relations.}$$

(again, better to take Borel construction). (forget BG action).

A_∞ category?: $\mathcal{F}(X, \mu, G, \text{simplified})$ object: $(L, \text{Spin str.})$.

$L \subset \mu^{-1}(0) \subset X$ G-equivariant curved; kill this using bounding co-chains.

Have:

$$CF^*(L_1, L_2) = \Omega(L_1 \cap L_2)/G \in \Lambda.$$

<u>Thm:</u> $\exists\ \mathfrak{b} \in H^*(Y; \Lambda_+)$ $\quad \Lambda_+ = \sum a_i T^{\lambda_i}$ $\quad \lambda_i > 0$, $a_i \in \mathbb{R}$,

such that

$$\mathcal{F}(X, \mu, G, \text{simplified})$$
$$\text{\rotatebox{90}{$\cong$}}$$
$$\mathcal{F}(Y, \mathfrak{b})$$

$\mathfrak{b}$ ← bulk deformation.

(Rmk: can put bulk on other side, too b/c exists a <u>Kirwan map</u> which is <u>surjective</u>).

compare
(c.f. Frauenfelder, Woodward).

(other references study
$$\begin{array}{ccc} X & \hookrightarrow & E \\ G & & \downarrow \uparrow u \\ & & \Sigma \end{array}$$
"Gauged σ-model", A-model studied by many people).

maybe related

<u>Rmk:</u> Way to calculate $\mathfrak{b}$ may involve ↑ ; the claimed Thm only claims $\mathfrak{b}$ is unique/exists, but not how to calculate it.

<u>Ex:</u> $\mathbb{C}^{n+k}$ $\circlearrowleft$ T^k $\qquad Y = $ toric manifold.

Consider $T^n \subset Y$. (orbit)

Have $x = (x_1, \ldots, x_n) \in H^1(T^n)$

& consider $\sum m_k^{\mathfrak{b}}(x, \ldots, x) = W(\mathfrak{b}, y)$, $y_i = e^{x_i}$

bulk deformed LG potential

thm: $\exists$ $\underline{\mathfrak{b}}$ so these match.

On the other hand $m_k^G(x, \ldots, x) = W^0(y)$

Leading order pot. fxn.

(in a sense, already true for toric mfolds by versions of FOOO; but this result more general).

Thm: can choose b, so

Idea: Use Lagrangian correspondence

$$L \subset \mu^{-1}(0) \subset X$$
$$\circlearrowright G.$$

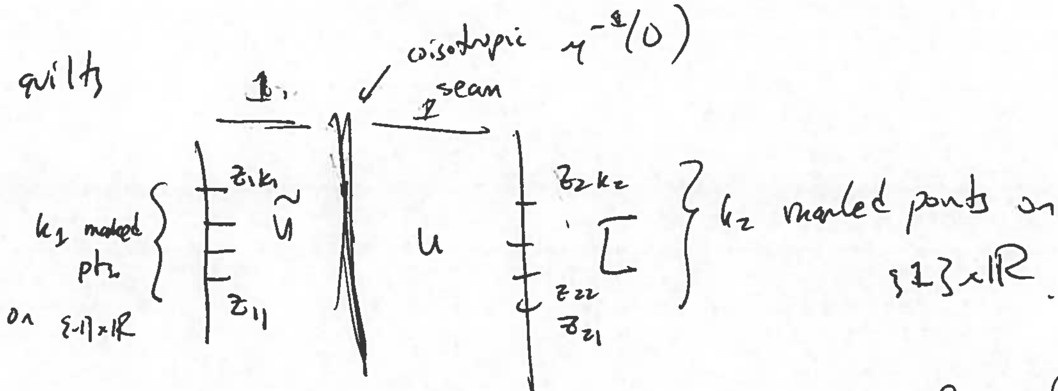
Then, $\bar{L} = L/G$. Then $(\Omega(L/G), m_k^G)$

$$(\Omega(\bar{L}), m_k)$$

$\exists$ a b s.t. $(\Omega(L/G), m_k^G)$ is unobstructed if $\Longleftrightarrow$

$$(\Omega(\bar{L}), m_k^b) \text{ is unobstructed.}$$

Study quilts



just assume energy finite, & don't worry about which asymptotics up & down.

$$\tilde{u} : [-1,0]\times\mathbb{R} \longrightarrow X$$

$$u : [0,1]\times\mathbb{R} \longrightarrow Y = X/\!\!/G$$

$$\tilde{u}(0,\tau) \in \mu^{-1}(0) \ \& \ \tilde{u}(0,\tau) = u(0,\tau)/G.$$

(recall $Y = \mu^{-1}(0)/G$).

$$\& \int \tilde{u}^* \omega_X \leq \infty$$
$$\int u^* \omega_Y < \infty.$$

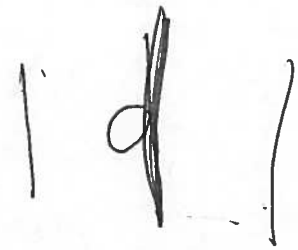
Have $\mathcal{M}_{k_1,k_2}(L, \bar{L})$ ← Rmk: G acts here

$$\begin{array}{ccc} & \mathcal{M}_{k_1,k_2}(L,\bar{L}) & \\ \swarrow & & \searrow \\ L^{k_1} & & \bar{L}^{k_2} \end{array}$$

G action:

$$G(\tilde{u}, u, z_1, z_2) = (g\tilde{u}, u, \vec{z}_1, \vec{z}_2)$$

Evaluation $(\tilde{u}, u) \xmapsto{ev_{-\infty}} \lim_{\tau \to -\infty} u(t,\tau) \in \bar{L}$



& have $\xmapsto[ev_{+\infty}]{}$.

$u^{-1}(0)$ $u: D^2 \to X \times Y$

Gives: $m_{k_1,k_2} : \Omega(\mathbb{L})^{\otimes k_1} \otimes \Omega(\bar{L}) \otimes \Omega(\bar{L})^{\otimes k_2} \longrightarrow \Omega(\bar{L} \otimes \Delta)$

Gives an A_∞ bimodule str; (as a grp) of $\Omega(\bar{L})$ over Ω

($HF(u_-^{-1}(0), L \times \bar{L}) =$)

bimodule ch. cplx?

(last year)

Lem: Say have: C_1, C_2 curved A_∞ alg.

& $C_1 \curvearrowright D \curvearrowleft C_2$ A_∞ bimodule & have $\mathbb{1} \in D$ "cycle element"

Then, C_1 is unobstructed $\iff$ C_2 is unobstructed

or rather (unobstr $\longleftrightarrow$ $\exists\, b$ w/ $m_k(b, \_, b) = 0$)

But this is wrong here; no need to find the bulk classes.

real Point: this is not an A_∞ bimodule unless you correct

RHS by a bulk term (A_∞ bimodule rel'ns not satisfied.

Recall a $C_1 - C_2$ bimodule D has operators $m_{k_1,k_2} : C_1^{k_1} \otimes D \otimes C_1^{k_2} \longrightarrow D$

satisfying ——

<insert>.

$\mathbb{1} \in D$ is a cycle elt. if:

(1) $m_{1,0}(\mathbb{1}) \equiv 0 \mod \Delta_+$.

can weaken: (as a left module, isomorphic to free, & as a right, isomorphic to free, up to error Δ_+.

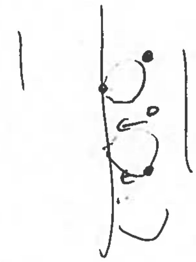
(2) $x \longmapsto m_{1,0}(x, \mathbb{1})$ $C_1 \to D$ vector space isomorphism

(3) $y \longmapsto m_{0,1}(\mathbb{1}, y)$ $C_2 \longrightarrow D$ is a vector space isomorphism.

To prove bimodule relation:

compactify space of maps, analyze its boundary.

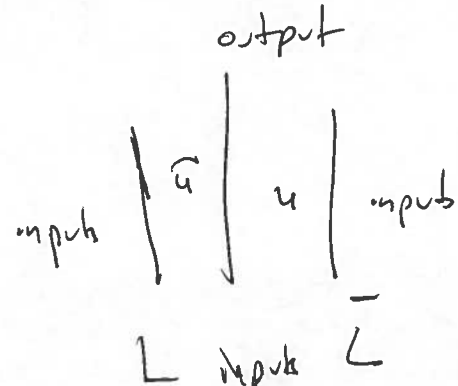
what is the boundary?



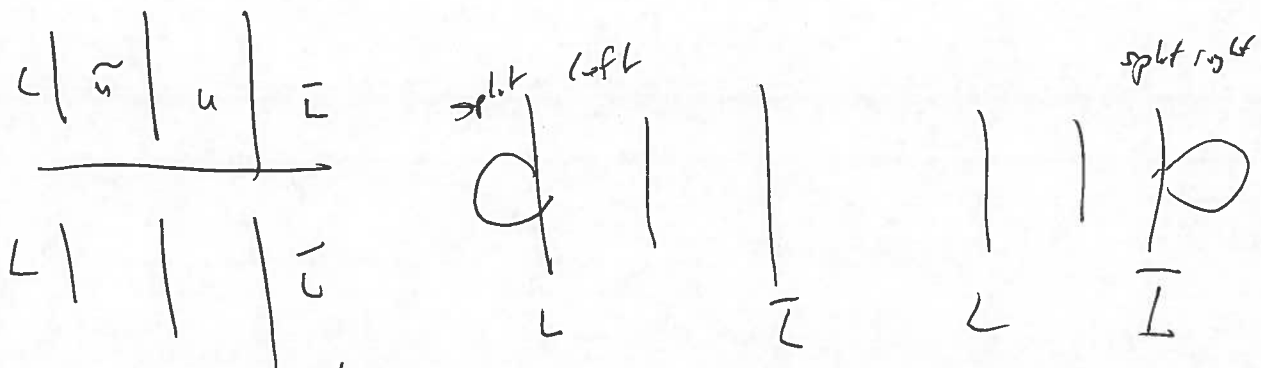
[Aside: recall:

$\mathcal{M}_{k_1,k_2}(L,\bar{L})$

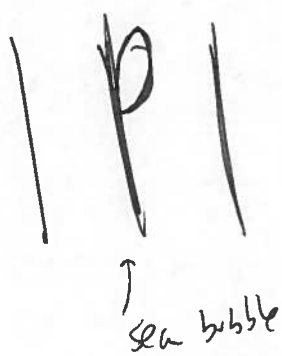
$\tilde{u}(-1,\tau)\in L,\ u(1,\tau)\in \bar{L}$



codim 1 boundary:



Interesting: not all the bubbling:



codim 2 bubble:

breaks bimodule relations

usually: $L_{12} \subset X_1^{-} \times \bar{X}_2 \rightsquigarrow W_{L_{12}} : \mathcal{F}(X_1) \to \mathcal{F}(X_2)$

$$\begin{array}{ccc} & L_{12} & \\ \swarrow & & \searrow \\ X_1 & & X_2 \end{array}$$

but need L_{12} be <u>unobstructed</u>.

Q: In toric case, easy to see b

(Fano case: b should be zero.

For us, $L_{12} = \{(x,y) \in X \times Y$ w/ $\psi(x) = 0,\ y = x \bmod G\}$

How to get L_{12} unobstructed?

use: <u>Thm</u>: (FOOO):

Suppose $L \subset M$

Then, $\exists$ $O_k(L) \in \dfrac{H^*(L;\Lambda_0)}{\operatorname{im}(H^*(M;\Lambda_0))}$

(look at proof: finding $\mathfrak{b}$, b very explicit)

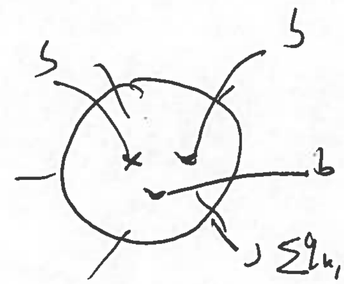
s.t.: If $O_k(L) \equiv 0$ for $k = 1, 2, 3, \ldots$

then, $\exists$ $\mathfrak{b} \in H^*(M)$, $b \in H^*(L)$,

s.t. $\sum m_k^{\mathfrak{b}}(b, \ldots, b) = 0$.

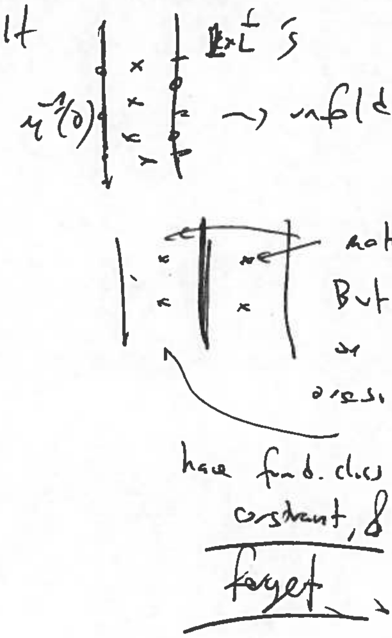
<u>Example</u>: $Y \hookrightarrow Y \times Y$ satisfies hypotheses of <u>Thm's</u> above

how to use $\mathfrak{b}$

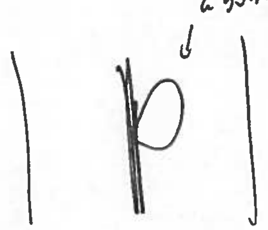


$\sum q_{k,\ell}(b, \ldots, b; x_1, \ldots, x_\ell)$

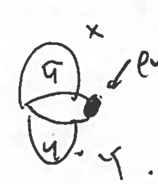
$\leftarrow$ do then insertions in <u>folded</u> quilt



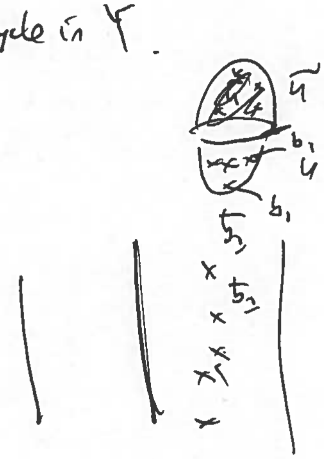
(eg. Kuranishi picture helps ensure obstruction classes are <u>equivalent</u>.



a bubble



evaluate here! get cycle in Y.

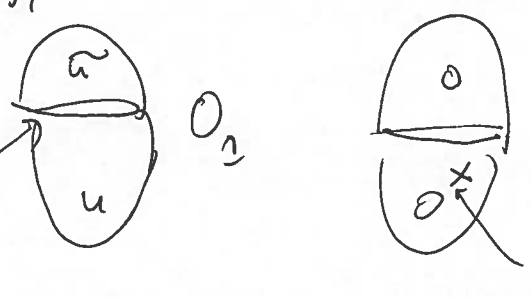


Smallest energy: cycle $\in$ ~~Y~~

gives $o_1 \in H_*(Y)$,

Take as: First term of $\mathfrak{b}$ is $\mathfrak{b}_1 = -o_1$

lowest energy



const map at ~~b~~ b_1, gives

$b_1 = -O_1$.

So, cancel to lowest energy, apply induction.

~~Now, ...~~ A_∞ Now, we check A_∞ bimodule relations.

to lift) is a non-formal G twist;

seems related to "twisted part of Gauged σ-model."

Generalized statement:

have $F: H_G^*(X) \xrightarrow{\text{non-linear map}} H^*(Y)$, Has property s.t. if take def

So that if put bulk class or below, the equivalent

$\tilde{b} \in H_G^*(X)$, then consider $\left(H_G^*(X), \cup^{\text{quantum}}_{b\text{-bulk}}\right) \xrightarrow[\text{ring homomorphism}]{\text{gives } D_b F \text{ (linear map)}} \left(H^*(Y, \cup^b)\right)$

quantum with deformed.

(※ seems crucial to know