$\hat{\Gamma}$-structures reading Group 1,

10/4/2016

J. Zhao, Introduction to the $\hat{\Gamma}_X$ class "Gamma-hat class"

Plan:

(I) Chern characters and Hirzebruch-R-R.

(II) Multiplicative sequences c.f. [Hirzebruch's book].

Using this, define $\hat{A}$-genus, Todd, Gamma-hat

(III) (time permitting) localization on $\mathcal{L}X$ & $\hat{\Gamma}_X$.

(I) Given a complex vector bundle E^r (rank) on a compact manifold M, then the total Chern class

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E) \underset{\text{by "splitting principle"}}{=} \prod_{i=1}^{r} (1+x_i),$$ x_i: Chern roots.

Prop: ["Splitting principle"], Given E,

$\exists$ a space, called $F\ell(E)$ (for "flag space") and a map $\sigma: F\ell(E) \to M$ s.t.

(1) $\sigma^*: H^*(M) \longrightarrow H^*(F\ell(E))$ is injective. (integrally, not just rationally)

(2) $\sigma^* E \cong L_1 \oplus \cdots \oplus L_r$, L_i line bundles

By naturality of Chern classes,

$$\sigma^* c(E) = c(\sigma^* E) \underset{\text{Whitney sum formula}}{=} \prod_{i=1}^{r} c(L_i) = \prod_{i=1}^{r} (1 + c_1(L_i))$$

Chern characters:

Define

$$ch(E) = \sum_{i=1}^{r} e^{x_i} = \sum_{n=1}^{r} \underbrace{\frac{1}{n!}(x_1^n + \cdots + x_r^n)}_{ch_n(E)}.$$

Can express in terms of Chern classes (using fact that Chern classes are symmetric polys in x_i).:

$$ch(E) = r + c_1 + \frac{c_1^2 - 2c_2}{2} + \cdots$$

induces a map $ch: K_0(M) \to H^*(M;\mathbb{Q})$.

Hirzebruch – Riemann – Roch theorem

Let E_1, E_2 be holo. vector bundles on a complex manifold M. There's an "Euler pairing"

$$\chi(E_1, E_2) = \sum_{i=0}^{\dim M} (-1)^i \dim \underbrace{\mathrm{Ext}^i(E_1, E_2)}_{\dim H^i(M, E_1^\vee \otimes E_2)} .$$

Hirzebruch RR:

$$\chi(E_1, E_2) = \int_M ch(E_1^\vee) . ch(E_2) . \underline{\underline{td}}_M$$

where td_M = Todd class of M. (to be defined)

& further, it will turn out there is an identity

$$(*) \qquad \underbrace{(2\pi i)^{\frac{\deg}{2}} td_M}_{} = e^{c_1(M)/2}\, \widehat{\Gamma}_M \cdot \widehat{\Gamma}_M^{*} \qquad \left(\text{where } \widehat{\Gamma}_M^{*} = \widehat{\Gamma}(TM^\vee)\right)$$

Shorthand notation meaning twist each term of td_M:

$$(2\pi i)^{\frac{\deg}{2}} td_M := \sum_{n=0}^{\dim M} (2\pi i)^n \, td_M^n .$$

(there's a similar twist of Chern character

$$(**) \qquad \mathcal{C}h_M := \text{"}(2\pi i)^{\deg/2} ch_M\text{"} \;).$$

td class : characteristic class associated to the power series $\frac{x}{1-e^{-x}}$;

meaning $\quad td(E) = \prod_{i=1}^{r} \frac{x_i}{1-e^{x_i}} \quad$ x_i Chern roots (of TM). $\quad td_M := td(TM)$

(recall the identity in complex analysis: $\frac{x}{1-e^{-x}} = e^{x/2}\, \Gamma\left(1 - \frac{x}{2\pi i}\right) \Gamma\left(1 + \frac{x}{2\pi i}\right)$)

Recall also that classically,

Euler's Gamma function $\quad \Gamma(z) := \int_0^\infty e^{-t} t^{z-1} \, dt \quad$, is well-defined on $Re(z) > 0$.

Identities: note • $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$
& (integration by parts):
• $\Gamma(z+1) = z\Gamma(z)$.
(in particular, $\Gamma(1+n) = n!$, $n \in \mathbb{Z}_{\geq 0}$)

can analytically continue & get a well-defined meromorphic function

on $\mathbb{C}$ w/ poles at negative ~~non-positive~~ real integers.

$$\widehat{\Gamma}_M := \prod_{i=1}^{r} \Gamma(1 + x_i)$$

← chern roots of TM.

Rmk: Γ is transcendental, so lands really in $H^\bullet(M; \underline{\mathbb{C}})$.

& (more generally, $\widehat{\Gamma}(E) = \prod_i \Gamma(1+x_i)$ ← chern roots of E.)

Paul: usually one sees the identity

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)}$$

(~~...~~ more usually, $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$), often called "Euler's reflection formula"

& a basic one: $\Gamma(1+x) = x\,\Gamma(x)$.

$$\text{So, } \Rightarrow \Gamma(1+x)\Gamma(1-x) = \frac{\pi x}{\sin(\pi x)} = \frac{2i\pi x}{e^{i\pi x}-e^{-i\pi x}} = \frac{1}{e^{\pi i x}}\left(\frac{2\pi i x}{1-e^{-2\pi i x}}\right).$$

Hence,

$$\Gamma\left(1+\frac{x}{2\pi i}\right)\Gamma\left(1-\frac{x}{2\pi i}\right) = \frac{x}{e^{x/2}-e^{-x/2}} = e^{-x/2}\left(\frac{x}{1-e^{-x}}\right).$$

Using (*) & (**), can rewrite HRR as:

$$\chi(E_1,E_2) = \left[\mathrm{Ch}(E_1)\hat{\Gamma}_M, \mathrm{Ch}(E_2)\hat{\Gamma}_M\right), \qquad (***)$$

where

$$[\alpha,\beta) := \frac{1}{(2\pi)^{\dim M}}\int_M \left(e^{i\pi c_1(M)}\, e^{i\pi\mu}\,\alpha\right)\cup\beta$$

$$\mu(\phi) = \left(p - \frac{\dim M}{2}\right)\cdot\phi, \quad \phi\in H^{2p}(M).$$

↑ another operator, where

(so expand $e^{i\pi\mu}$ in exponential terms, & once know degree of α, apply).

(will apply μ to α)!

Paul: to derive (***):

$$\mathrm{Td}(TM) = e^{\frac{c_1(M)}{2}}(2\pi i)^{-\frac{\deg}{2}}\left(\Gamma(TM^*)\Gamma(TM)\right)$$

cf note this is correct, not (*)

HRR:

$$\langle E_1,E_2\rangle = \int_M \mathrm{Td}(TM)\,\mathrm{ch}(E_1^*)\,\mathrm{ch}(E_2)$$

$$= \int_M e^{\frac{c_1(M)}{2}}(2\pi i)^{-\frac{\deg}{2}}\left(\Gamma(TM^*)\Gamma(TM)\right)\mathrm{ch}(E_1^*)\,\mathrm{ch}(E_2).$$

$$= \int_M e^{\frac{c_1(M)}{2}} \left((2\pi i)^{-\frac{\deg}{2}} \left(\Gamma(T\cancel{M}^*)\right) ch(E_1^*) \right) \cdot \left((2\pi i)^{-\frac{\deg}{2}} \left(\Gamma(TM)\right) ch(E_2) \right).$$

$$= (2\pi i)^{-\dim_{\mathbb{C}} M} \int_M e^{\pi i c_1(M)} \Gamma(TM^*) (2\pi i)^{\deg/2} ch(E_1^*) \Gamma(TM) (2\pi i)^{\deg/2} ch(E_2).$$

mult. out by this & inside by $(2\pi i^{-\deg/2})$

(when integrate over M, only $\deg = 2\dim_{\mathbb{C}}$ survives!)

Compare this to formula (***)

(which is from [Galkin-Iritani])

(note: there may be some differences?)

(II) Multiplicative sequences

Def'n: Let ~~A sequence~~ $\{K_i(p_1, \ldots p_i)\}$ be a sequence of polynomials in p_i's.

We call it a multiplicative sequence if, whenever there exists a factorization

$$(1 + p_1 z + p_2 z^2 + \cdots) = (1 + p_1' z + p_2' z^2 + \cdots)(1 + p_1'' z + p_2'' z^2 + \cdots)$$

then $\Rightarrow$
$$\sum_{j=0}^{\infty} K_j(p_1, \ldots, p_j) z^j = \left(\sum_{j=0}^{\infty} K_j(p_1', \ldots p_j') z^j\right)\left(\sum_{j=0}^{\infty} K_j(p_1'', \ldots, p_j'') z^j\right)$$

evaluating at $(p_1, p_2, \ldots) = (1, 0, \ldots, 0)$; & denote $b_i = K_i(1, \underbrace{0, \ldots, 0}_{i-1})$

w/ $b_0 = 1$.

$$\Rightarrow \sum_{j=0}^{\infty} K_j(p_1, \ldots, p_j) z^j$$

gives

$$Q(z) = \sum_{i=0}^{\infty} b_i z^i \quad \text{w/ } b_0 = 1.$$

Prop: There is a bijection

$$\{\text{multiplicative sequences}\} \longleftrightarrow \left\{\begin{array}{c}\text{power series w/ constant term} \\ \text{equal to } 1\end{array}\right\}$$

(& recursive algorithm to go this way!) Recipe:

Given a formal power series $Q(z)$,

~~take~~ solve: $\prod_i Q(x_i) = 1 + \sum K_j(p_1, \ldots, p_j)$

where p_i is the ith elementary symmetric polynomial of x_i

& each K_j should be homogeneous of degree $2j$, where each $|x_i| = 2$ and thus $|p_i| = 2i$.

→ (Get multiplicative characteristic classes of cplx. vector bundles, by taking $K_i(c_1(E), \dots, c_i(E))$ for all i.)

Ex:

(1) Todd class Td

is derived from
$$\frac{z}{1-e^{-z}} = 1 + \frac{1}{2} z + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k}$$
B_k: k^{th} Bernoulli numbers.

(2) Gamma-hat class $\hat{\Gamma}_M$ comes from
$$\Gamma(1+z) = \exp\left(-\gamma z + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} z^k\right),$$
ζ: zeta function

γ: some constant, "Euler constant"

Rmk: γ (expected to be transcendental, but not even known to be irrational!)

First few terms:

$\hat{\Gamma}^0_M = 1$ $\quad \hat{\Gamma}^1_M = -\gamma c_1$

$$\hat{\Gamma}^2_M(c_1, c_2) = \frac{1}{2}\gamma^2 c_1^2 + \frac{1}{2}\zeta(2)(c_1^2 - 2c_2).$$

$$\hat{\Gamma}_M := \exp\left(-\gamma c_1 + \sum_{k=2}^{\infty} (-1)^k (k-1)!\, \zeta(k)\, ch_k(M)\right)$$

$(k-1)!$: "$\frac{k!}{k}$"

$ch_k(M)$: k^{th} Chern char. assoc. to M.

Recapping/
Continuing the earlier discussion, let's note that the existence of $\hat{\Gamma}_M$ (which is roughly a 'square root' of Td_M, up to factors), can allow one to re-express HRR in the following nice form:

"a corrected (by $\hat{\Gamma}_M$) Chern character map (up to factor/sign) intertwines the Euler pairing and the integration pairing (also corrected by factors/signs)."

Specifically

(Let our complex manifold be called X now, instead of M.)

Thm [HRR reinterpreted]: If $\langle E_1, E_2\rangle$ denotes the Euler pairing $\chi(E_1,E_2) = \chi(Ext^*(E_1,E_2))$, then

$$\boxed{\langle E_1, E_2\rangle = [\hat{\gamma}(TX)ch(E_1),\ \hat{\gamma}(TX)ch(E_2)]},$$

where
- $\hat{\gamma}(TX) = (2\pi i)^{-\frac{deg}{2}}\ \hat{\Gamma}(TX)$. (this is the class assoc. to the series $\gamma(z) = \Gamma(1+\frac{z}{2\pi i})$.)

  "degree operator /2" ($deg := deg(-)\cdot id$).
- $[\alpha, \beta] = \int_X e^{c_1(X)/2}\left[(-1)^{deg/2} x_1\right] x_2$.

Rmk: Note that both $\langle -,-\rangle$ & $[-,-]$ are (graded) (anti-)symmetric in the Calabi-Yau case, the former by using Serre duality. $\rightarrow$ "corrected Chern character is an 'isometry'."

Proof: (starting from usual HRR):

Using the definition of Td & Γ, & the identity

$$\Gamma(1+x)\Gamma(1-x) = \frac{1}{e^{\pi i x}}\left(\frac{2\pi i x}{1-e^{-2\pi i x}}\right)$$

implies $\hat{\Gamma}(E)\hat{\Gamma}(E^*) = 2\pi i^{deg/2}[Td(E)]e^{-\pi i c_1}$

assoc. to $\Gamma(1-x)$ b/c the Chern roots changed sign

$\frac{2\pi i x}{1-e^{-2\pi i x}}$ instead of $\frac{x}{1-e^{-x}}$.

Equivalently,

$$Td(E) = (2\pi i)^{-deg/2}\left(\hat{\Gamma}(E)\hat{\Gamma}(E^*)\right)e^{\pi i c_1}$$
$$= 2\pi i^{-deg/2}\left(\hat{\Gamma}(E)\hat{\Gamma}(E^*)\right)e^{c_1(E)/2}$$

Thus, HRR $\Rightarrow$

$$\langle E_1, E_2\rangle = \int_X Td(TX)ch(E_2)ch(E_1^*)$$
$$= \int_X (2\pi i)^{-deg/2}\left(\hat{\Gamma}(TX)\hat{\Gamma}(TX^*)\right)e^{c_1(X)/2} ch(E_2)ch(E_1^*).$$

rearranging,

$$= \int_X e^{c_1(X)/2}\left(ch(E_1^*)\hat{\gamma}(TX^*)\right)\left(ch(E_2)\hat{\gamma}(TX)\right)$$
$$= \int_X e^{c_1(X)/2}\left[(-1)^{deg/2}\left(ch(E_1)\hat{\gamma}(TX)\right)\right]\left(ch(E_2)\hat{\gamma}(TX)\right).\ \blacksquare$$