J. Zhao, Loop space interpretation of $\hat{\Gamma}_X := \hat{\Gamma}(TX)$ 10/11/2016

Recap: Given a complex v.b. $E^n \to X$ ← cplx. manifold,

have $\underset{\text{total chern class}}{c(E)} = \prod_{i=1}^{n}(1+x_i)$ ← Chern roots (formal decomp, using splitting principle).

$c_i(E)$ = elementary sym. polynomials in x_i's.

Using this formal point of view, can write down other char. classes as symm. fns. of x_i ($\Rightarrow$ functions of $c_1, \ldots, c_n$).

For instance, define $Td(E) = \prod_{i=1}^{n}\frac{x_i}{1-e^{-x_i}} = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1^2(E) + c_2(E)) + \cdots$

↑ series expansion, write in terms of c_i's.

Similarly, $\hat{\Gamma}(E) = \prod_{i=1}^{n}\Gamma(1+x_i) = \cdots$

& A-roof genus $\hat{A}(E) = \prod_{i=1}^{n}\frac{x_i/2}{\sinh x_i/2}$ (using the fact $\frac{z/2}{\sinh z/2} = 1 - \frac{1}{24}z^2 + \frac{7}{5760}z^4 - \cdots$ is an even function, so when plug in Chern classes, get $4k$ classes, hence, b/c its even, can rewrite in terms of Pontryagin classes (of a "real part", if $E = E_{\mathbb{R}} \otimes \mathbb{C}$) p_i by using the polynomial $\frac{\sqrt{x}/2}{\sinh\sqrt{x}/2}$ (take square roots).

Shorthand: $\hat{A}_X := \hat{A}(TX)$.

Note: $\hat{A}_X$ is defined for real smooth manifolds.

& if $\dim X = 4n$ & X is Spin,

then Thm/Fact: $\hat{A}_X([X]) \in \mathbb{Z}$, $\hat{A}_X([X])$.

Ex: $X = \mathbb{P}^n$. There is a SES:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}(1)^{n+1} \to T\mathbb{P}^n \to 0$$

$\Rightarrow$ b/c $\mathcal{O}_{\mathbb{P}^n}$ is trivial,

$$Td(T\mathbb{P}^n) = Td(\mathcal{O}(1)^{n+1}) \underset{\substack{Td \text{ is} \\ \text{multiplicative}}}{=} (Td(\mathcal{O}(1)))^{n+1} = \left(\frac{a}{1-e^{-a}}\right)^{n+1},$$

where $a = c_1(\mathcal{O}(1))$.

Rmk: The identity $\frac{x}{1-e^{-x}} = e^{x/2}\frac{x/2}{\sinh x/2}$,

$\Rightarrow \boxed{Td(E) = e^{c_1(E)/2}\hat{A}(E)}$ (for a cplx. vec. bdle.).

(I) Loop spaces

Given a complex manifold X^n, consider $E = TX$.

$\mathcal{L}X = \mathrm{Maps}(S^1, X)$, & have a canonical inclusion of constant loops

$i: X \hookrightarrow \mathcal{L}X$. The normal bundle to this

$$N_{\mathcal{L}X/X} := \coprod_{x_0 \in X} \{ \gamma : S^1 \to T_{x_0}X \} \Big/ T_{x_0}X \quad \text{(constant loops)}.$$

(or concretely, "maps whose average is zero").

$$\begin{array}{c} N_{\mathcal{L}X/X} \\ \downarrow \\ X \end{array}$$

Fixing a point $x_0 \in X$, $T_{x_0}X \cong \mathbb{C}^n$, & the fibers $(N_{\mathcal{L}X/X})_{x_0} \cong \mathrm{Maps}(S^1, \mathbb{C}^n)/\mathbb{C}^n$

Given $f : S^1 \to \mathbb{C}^n$, f admits a Fourier decomposition

$$f \rightsquigarrow \begin{pmatrix} \sum_{k \in \mathbb{Z}} c_k^1 e^{2\pi i k t} \\ \vdots \\ \sum_{k \in \mathbb{Z}} c_k^n e^{2\pi i k t} \end{pmatrix}$$

& can arrange that all constant terms are zero (c_0^i), by

$$\in \mathbb{C}[[z, z^{-1}]] \otimes \mathbb{C}^n, \quad \text{where } z = e^{2\pi i t}.$$

There is an S^1-action on $\mathcal{L}X$ acting by $\theta \cdot g(t) = g(t+\theta)$,

which restricts to an action on $N_{\mathcal{L}X/X}$ (b/c X is fixed); on

a fixed k, in Fourier decomposition, get

$$c_k^i z^k \xrightarrow{\cdot \theta} c_k^i e^{2\pi i k \theta} \cdot z^k$$

$\Rightarrow$ induced action on Fourier coefficients.

It follows that (formally, at least) & looking over all points x_0 now

$$N_{\mathcal{L}X/X} \text{"}:=\text{"} \bigoplus_{\substack{k \in \mathbb{Z} \\ k \neq 0}} TX \otimes \eta^k$$

where η^k = complex 1-dim'l rep'n of S^1 of weight k.

Now we can define (& $\mathcal{N}_-$ similarly, $\mathcal{N}_- = \bigoplus_{k<0} - -$)

$$\mathcal{N}_+ := \bigoplus_{k=1}^{\infty} TX \otimes \eta^k$$, & note that

$$\boxed{N_{\mathcal{L}X/X} = \mathcal{N}_+ \oplus \mathcal{N}_-}$$

(we should think of $\mathcal{N}_+$ loops as "all loops that can be the boundary of a hol. curve.*"). (recall that we're on a cplx. manifold).

(Heuristic)

Atiyah-Witten: $\dfrac{1}{e_{S^1}(N_{\mathcal{L}X/X})}$ is a "u-defined $\hat{A}_X$."

It turns out that $\dfrac{1}{e_{S^1}(\mathcal{N}_+)}$ is (roughly) a "u-defined $\hat{\Gamma}_X$".

(II) Equivariant Euler class

Given $S^1 \curvearrowright X$, X fin. dim'l mfold.

$H^*_{S^1}(X) := H^*(X \times_{S^1} ES^1)$. $\exists_1$ canonical S^1-equiv. map $f: X \to pt.$, $\sim_{S^1}$ $\sim_{S^1}$ triv., inducing ($ES^1 = S^\infty$)

$f^*: H^*_{S^1}(pt) \to H^*_{S^1}(X)$, gives a $\mathbb{Z}[u]$-structure.

$H^*_{S^1}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[u]$, $|u| = 2$.

Let F denote the fixed point set.

There is an S^1-equiv. inclusion map

$i: F \hookrightarrow \cancel{X}$ inducing $i^*: H^*_{S^1}(\cancel{X}) \to H^*_{S^1}(F)$.

If k denotes the codim of $N_{X/F}$,

note that there is also a wrong-way (integration) map i_*.

defined via $i_*: H^{*-k}_{S^1}(F) \xrightarrow[S^1\text{-equiv.}]{} H^*_{S^1}(X, X \setminus F) \xrightarrow{j^*} H^*_{S^1}(M)$.

Thom iso.

Define $e_{S^1}(N_{X/F}) := i^* i_* 1$.

Recall: By Chern-Weil theory, for $E := N_{X/F}$,

if R_E = the curvature 2-form of E with respect to a Hermitian metric,

$$e(E) = \det\left(\frac{R_E}{2\pi i}\right)$$

(note if $E \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, then $R_E = \begin{pmatrix} 2\pi i c_1(L_1) & & \\ & \ddots & \\ & & 2\pi i c_1(L_n) \end{pmatrix}$ & can think of these more generally as eigenvalues, or apply splitting principle).

Equivariantly, a similar formula (corrected) holds:

$$e_{S^1}(E) = \det\left(\frac{\underset{\text{deg. 2}}{u} L_E + R_E}{2\pi i}\right),$$

(this is the curvature for the equivariant normal bundle; measures parallel transport in $\mathbb{CP}^\infty$ direction in the Borel construction).

where L_E is the matrix associated to the infinitesimal vector field X generated by S^1 action.

e.g. $S^1 \curvearrowright \mathbb{P}^1$

$N := N_{\mathbb{P}^1/p_1}$,

$L_N := 2\pi i \cdot$

Apply all of this to $\mathcal{N}_+ := \bigoplus_{k\geq 1}^{\infty} TX \otimes \eta^k$;

but problem: $\mathcal{N}_+$ is ∞-dim'l. Ad hoc argument (from physics):

take a finite-dim'l approximation, (& take a "limit"):

$$\mathcal{N}_{+,d} := \bigoplus_{k=1}^{d} TX \otimes \eta^k \circlearrowright S^1.$$

The fixed points are just X; $F = X$. Apply the above reasoning b/c know weight of S^1 action is k on $TX \otimes \eta^k$.

$$\text{“}\, e_{S^1}(\mathcal{N}_{+,d}) = \prod_{k=1}^{d} \det\left(\frac{u\, 2\pi i k + R_{TX}}{2\pi i}\right) \text{”}$$

(turn)

& define $\boxed{e_{S^1}(\mathcal{N}_+) \text{ “}=\text{” } \lim_{d\to\infty} e_{S^1}(\mathcal{N}_{+,d})}$ (not quite, it will turn out this diverges & need to regularize).

Compute:

$$e_{S^1}(\mathcal{N}_+) = \prod_{k=1}^{\infty} \det\left(uk\left(\mathrm{Id} + \frac{R_{TX}}{2\pi i uk}\right)\right).$$

(By splitting principle, as R_E "=" $\begin{pmatrix} 2\pi i c_1(L_1) & & \\ & \ddots & \\ & & 2\pi i c_1(L_r) \end{pmatrix}$,

it follows
$\det\left(\mathrm{Id} + \frac{R_E}{2\pi i}\right)$ is total chern class.)

$$= \prod_{k=1}^{\infty} (uk)^{\dim X} \prod_{k=1}^{\infty} \det\left(\mathrm{Id} + \frac{R_{TX}}{2\pi i uk}\right)$$

→ eigenvalues are $2\pi i x_j$, so $2\pi i$'s cancel.

Chern roots.

$$= \underbrace{\prod_{k=1}^{\infty} (uk)^{\dim X}}_{1^{st}\text{ term}} \underbrace{\prod_{k=1}^{\infty} \prod_{j=1}^{\dim X} \left(1 + \frac{x_j}{ku}\right)}_{2^{nd}\text{ term.}}$$

Warning! Both terms are divergent. . To deal with this, look at regularized $e_{S^1}(\mathcal{N}_+)$.

1st term: Use zeta function regularization:

Recall: $\underline{u} := \{u_n\}$ a sequence $\rightsquigarrow$ zeta function $\underbrace{Z_{\underline{u}}(s) = \sum_{n=1}^{\infty} u_n^{-s}}_{\text{regularizes the sum}}$.

For a product: if $Z_{\underline{u}}(s)$ has an analytic continuation, which is holomorphic at $s=0$, then regularize

$$\text{"}\prod_{n=1}^{\infty} u_n\text{"} := \exp\left(-\frac{\partial}{\partial s} Z_{\underline{u}}(s)\Big|_{s=0}\right).$$

$$= \exp\left(+\sum_n u_n^{-s} \ln u_n\Big|_{s=0}\right) = \exp\left(\sum_n u_n\right).$$

Using the zeta function regularization, can compute

$$\prod_{k=1}^{\infty} (uk)^{\dim X} = \left(\sqrt{\frac{2\pi}{u}}\right)^{\dim X}.$$

(calculation uses $\zeta(0) = \frac{1}{2}$, $\zeta'(0) = -\log\sqrt{2\pi}$)

<u>2nd term:</u> regularized determinant:

$$\det_{reg}(1+A) := \det\left((1+A)e^{-A}\right)$$

(has eigenvalues which are decaying/damped versions of prev. eigenvalues $\rightsquigarrow$ helps convergence!)

Using this, get

$$e_{S^1}(\mathcal{N}_+) = \left(\frac{2\pi}{u}\right)^{\frac{\dim X}{2}} \prod_{k=1}^{\infty} \det_{reg}\left(Id + \frac{R_{TX}}{2\pi i k u}\right).$$

(splitting principle)
$$= \left(\frac{2\pi}{u}\right)^{\frac{\dim X}{2}} \prod_{k=1}^{\infty} \prod_{j=1}^{\dim X} \left(1 + \frac{x_j}{ku}\right) e^{-\frac{x_j}{ku}}.$$

<u>Fact</u> (<u>Weierstrass form</u> of Γ):

know

$$e^{\gamma z}\Gamma(1+z) = \left[\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right]^{-1}.$$

Setting $z = \frac{x_j}{u}$: $\Rightarrow$

$$\left(e_{S^1}^{reg}(\mathcal{N}_+)\right)^{-1} = \left(\frac{u}{2\pi}\right)^{\frac{\dim X}{2}} e^{-\gamma\frac{c_1}{u}} \prod_{j=1}^{m} \Gamma\left(1+\frac{x_j}{u}\right)$$

Recall there was the operator $\mu(\phi) = \left(\frac{\dim}{2} - p\right)\phi$ $\phi \in H^{2p}$; these signs appear in clearing out u's.

If multiply by $\left(e^{reg}_{S^1}(N_-)\right)^{-1}$, get $\hat{A}\ \left(e^{-\gamma \frac{c_1}{4}}\ \right)$.