D. Treumann, Connections w/ irregular singularities & Stokes data 10/17/2016

X closed Riem. surface

$\cup$

S a finite set, &

E an algebraic vector bundle on $X \setminus S$.

(on $X \setminus S$), no "GAGA", so E with its algebraic structure is more information than the underlying holomorphic bundle.

It gives notion of algebraic sections of E.

In particular, near punctures, can discuss sections of 'polynomial growth.': meaning

If $e \in \Gamma(\mathcal{E})$, $|e(z)|$ grows like r^N as $r = |z| \to s \in S$.

(why need "algebraic"?

On tor. line bdle

$$\begin{tikzcd} V \arrow[rr, "e^{z}\cdot", "\cong"'] \arrow[dr] & & V \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

$\leftarrow$ multiply by $\exp(z)$.

messes w/ notion of "polynomial growth" at ∞)

Today:

We'll study algebraic connections on E, which is a map

$$\nabla : E \longrightarrow \Omega^1_{X \setminus S} \otimes E$$

algebraic, & obeys $\nabla(fe) = f\nabla e + df \otimes \nabla f$.

Can take the kernel E^{∇} in the analytic topology.

It's a local system, with fiber $(E^{\nabla})_{x_0}$ = germs of holomorphic (not nec. algebraic) sections of E near x_0 that are ∇-flat.

By analytic continuation, get a map:

$$\left\{ \begin{matrix} \text{rank } n \\ (E, \nabla) \\ \text{an algebraic} \\ \text{connection} \end{matrix} \right\} \overset{(*)}{\longrightarrow} \left\{ \begin{matrix} \text{Reps of} \\ \pi_1(X-S, x_0) \to GL(E^{\nabla}_{x_0}) \\ \qquad\qquad \cong \\ \qquad\qquad GL(\mathbb{C}^n) \end{matrix} \right\}$$

In coordinates, letting z be a uniformizer near x_0.

$E \cong \mathcal{O}^n$ (choose some basis); can always write

$\nabla = d - A\,dz$, where $A = A(z)$ is a matrix-valued holomorphic function near x_0. (will even be algebraic if we used algebraic coordinates)

&

$$E_{x_0}^{\nabla} := \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \text{ obeying } \begin{pmatrix} s_1' \\ \vdots \\ s_n' \end{pmatrix} = A \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}.$$

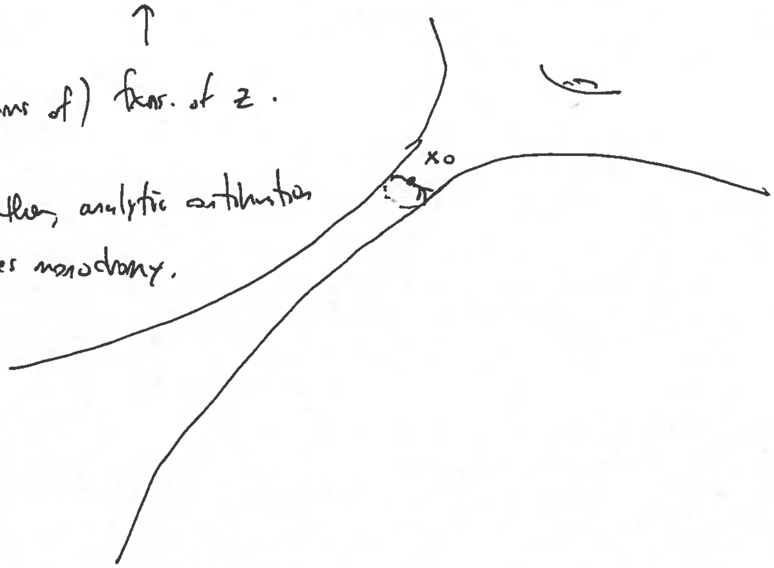
stalk

(& Taylor expansion or existence/uniqueness ODE results tell us rank is n)

$\uparrow$

(germs of) fcns. of z.

and then analytic continuation gives monodromy.



If S is $\emptyset$, the map $(*)$ from (E, ∇) to $\mathrm{Reps}(\pi_1(X \setminus S, x_0), GL(E_{x_0}^{\nabla}))$ is a bijection.

If $S \neq \emptyset$, it's <u>not</u> a bijection.

<u>Def</u>: (Deligne, 1970):

Say (E, ∇) has regular singularities at $s \in S$ if all solutions $e \in E^{\nabla}$ have moderate (meaning <u>polynomial</u>) growth as $z \to s$.

[Equivalently, if in coordinates near s, $\nabla = d + \left(\frac{A_{-1}}{z} + A_0 + A_1 z + \cdots\right)dz$ where $z = 0$ at s.].

<u>Thm</u> (Deligne 1970): [cf. Lecture notes in Math, vol. 163]

$$\begin{tikzcd}
\left\{\begin{array}{c}(E,\nabla) \text{ w/} \\ \text{regular singularities}\end{array}\right\} \arrow[d, phantom, "\cap"] \arrow[dr, "\cong"] & \\
\left\{\begin{array}{c}\text{rank } n \\ (E,\nabla) \\ \text{an alg. connection}\end{array}\right\} \arrow[r, "(*)"] & \left\{\begin{array}{c}\text{Reps of} \\ \pi_1(X\setminus S, x_0) \longrightarrow GL(E_{x_0}^{\nabla}) \\ \cong \\ GL(\mathbb{C}^n)\end{array}\right\}
\end{tikzcd}$$

(If $S=\phi$, pf. would be GAGA if ∇ was $\mathcal{O}$-linear; this is GAGA $+\,\varepsilon$!

If (E,∇) does <u>not</u> have regular singular points, say it's "irregular."

Map $(*)$ is surjective, & Q: what info is lost by taking E^{∇}?

Answer in two stages:

- Levelt/Turrittin theory of <u>formal</u> ODEs.
- Deligne's theory of Stokes structures.

Restrict attention to

$$(**)\left\{\begin{array}{l} X = \text{Riemann sphere} \\ S = \{0, \infty\} \\ (E,\nabla) \text{ has a regular singularity at } 0 \\ \qquad\qquad \text{irregular at } \infty. \end{array}\right.$$

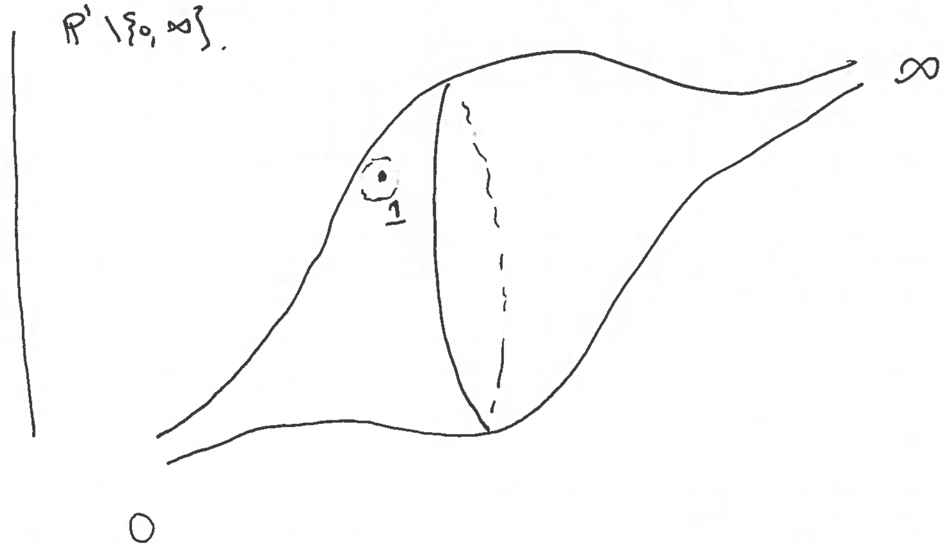
(Rmk: need to allow a reg. sing. at 0 in order to get monodromy)

(Rmk: in the literature, often the irreg. singularity is at 0).

$(**)$ is No loss of generality: "Birkhoff extension theorem" says that ~~the~~ data above$^{(**)}$ is <u>equivalent</u> to the data of a germ of a meromorphic ODE around ∞. [Reference?]

(analytic nhood, e.g., convergent series, <u>not</u> a formal nhood).

(cont'd on next page).



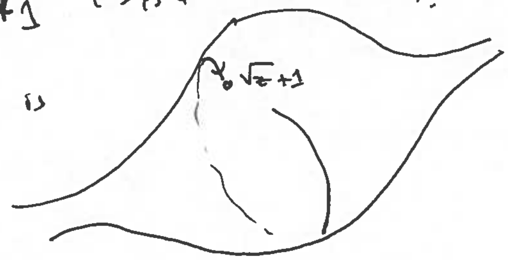
Recall that there's a path:

$$\begin{array}{ccc} & (E,\nabla) \text{ where } E \text{ has rank } n & \\ \text{easy} \begin{pmatrix} f \\ f' \\ \vdots \\ f^{(n-1)} \end{pmatrix} \uparrow & & \downarrow \text{ "cyclic vectors" hard.} \\ & f^{(n)} + a_{n-1} f^{(n-1)} + \dots + a_0 f = 0 & \end{array}$$

ODE (on $\mathbb{P}^1$ w/ canon. z-coord).

Hence: to give on $\mathbb{P}^1$ an (E,∇) of rank n, it suffices to give a function in a nbhd of 1 that obeys an nth order ODE. ("which is rational in z," built out of rat'l fns.) (generic function doesn't obey these)

Ex: "$\sqrt{z}+1$" (obeys a 2nd order ODE).

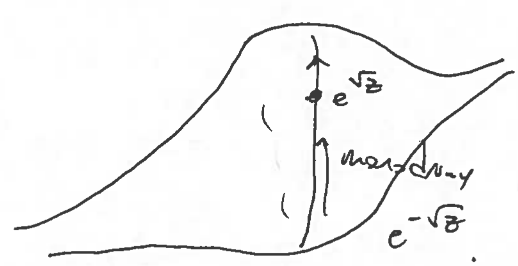


monodromy matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

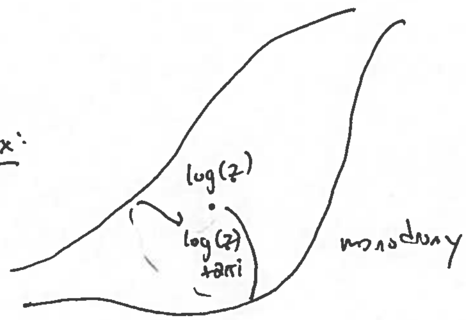
(maybe in some basis? not quite..)

Ex: $e^{\sqrt{z}}$ obeys a 2^{nd} order ODE.



as $z \to 0$ regular growth, as $z \to \infty$, irregular growth

Ex:



monodromy $\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$.

Levelt: Study (E, ∇) by studying its formal completion at ∞.

z := usual coordinate in $\mathbb{P}^1$.

Have $\mathbb{C}((z^{-1}))$, (polynomials + ∞-Laurent tail)

is a "differential field" (field, + can differentiate elts.)

(rather $(\mathbb{C}((z^{-1})), \frac{d}{dz})$ is a differential field).

The completion of E at ∞ is (w.r.t. valuations on analytic nhood of ∞).

$\hat{E}$, an n-dimensional vector space over $\mathbb{C}((z^{-1}))$,

equipped w/ $\hat{\nabla}$ connection satisfying Leibniz: $\hat{\nabla}(fe) = f\hat{\nabla}(e) + f' \cdot e$

formal connection

(theory of these is maybe called Differential Galois theory)

Levelt – Turrittin – (Hukuhara?)

They make an abelian category where every object has finite length.

- The one-dimensional modules (all irreducible) come from functions of the form

a mod $\mathbb{Z}$ is unique.

z^{a} gives monodromy

$$z^{p_0} e^{\underbrace{p_1 z + p_2 z^2 + \cdots + p_n z^n}_{\text{unique}}}$$

(such a function obeys a 1st order ODE, hence gives a formal connection)

& every one-dim. module is formally equivalent to one of these.

- Every irreducible $(\hat{E}, \hat{\nabla})$ is equivalent to a pushforward from a finite cover $\mathbb{C}((z^{-1/k}))$ of a 1-dimensional module, i.e. it comes from

"monodromy"

$$z^{p_0} \exp\left(p_{1/k} z^{1/k} + p_{2/k} z^{2/k} + \cdots + p_{n/k} z^{n/k}\right).$$

- Every indecomposable has only one (type of) simple in its Jordan-Hölder series, or rather, all simples occurring are isomorphic

($\Leftrightarrow$ no non-trivial Exts between different types).

• (these look like $\log(z)^2 z^{a} \cdot \exp\left(\text{——} \right)$) ..
(e.g.)

So, from (E, ∇) on $\mathbb{P}^1$ as before, ~~we can extract~~ can write

$$(\hat{E}, \hat{\nabla}) = \bigoplus_{a} \text{Indecomp.} \qquad \hat{D}^*$$

each one has $\qquad$ $k_a : 1$ cover

$$p_a(z^{1/k_a}) : \hat{D}_a^* \longrightarrow \mathbb{C}$$

In other words, if $\hat{D}^*$ is the punctured formal disk at ∞, we can extract (from (E, ∇)) a "curve" inside $\hat{D}^* \times \mathbb{C}$

(union of all graphs, think of it as a multiple valued function.

~~Although this is a formal curve~~

Now, Note $X := \mathbb{P}^1 \setminus \{0, \infty\}$ has a canonical "bordification" $X_{\log(S)}$ & a canonical extension of E.

replaces $\mathbb{P}^1 \setminus \{0, \infty\}$ by

$(E^{\nabla}_{\text{boundary point } \theta})$ = solutions to $\nabla e = 0$ in a "sector."

S^1 near ∞

S^1 near 0.

(not quite real blow-up, b/c want $z \longmapsto z^2$ to induce map of bordifications.
(but homeomorphic)).

In our formal case, similarly have

Rmk: this doesn't know the monodromy of the connection

$$\begin{array}{ccc} \text{Levelt curve} & \longrightarrow & \coprod_a S^1_a \\ \downarrow & & \downarrow \\ \hat{D}^* & \rightsquigarrow & S^1 \end{array}$$

log boundary

from "bordification"

Deligne defines a partial order on the fibers of this map. (by order of growth)

Let's see this in an example:

$$-\frac{\hbar^2}{2}\psi'' + \frac{x^2\psi}{2} = E\psi \qquad \text{(quantum) Harmonic oscillator.}$$

At $E = \frac{\hbar}{2}$ (ground state)

$\psi = e^{-x^2/2\hbar}$ is one solution (physically, this is the most relevant)

other solution:

$$\psi = \frac{\int_0^x e^{t^2/\hbar}\,dt}{e^{x^2/2\hbar}} \qquad \psi \sim e^{x^2/2\hbar} \text{ as } x \to \pm\infty$$

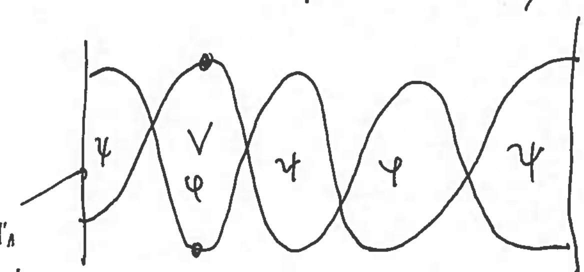
(so not a wave function)

⇝ the Levelt polynomials

$$P_a = \frac{x^2}{2\hbar}, \quad P_b = \frac{-x^2}{2\hbar}.$$

Thinking now of x as a complex parameter: let's draw $\coprod S^1_a$:

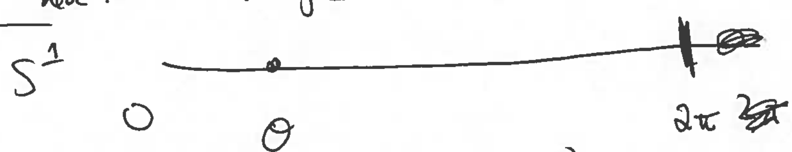
$S^1_a \amalg S^1_b$



Re $\left(\pm(re^{i\theta})^2/2\hbar\right)$ for $r \gg 0$, constant, $\hbar$ const., as fn. of θ.

(writing $x = re^{i\theta}$ for r large, we've drawn their real parts).

the sol'n w/ exponential decay as we head to ∞ in this angle



Partial order is a total order (by height) away from finitely many points in the base (stokes rays or anti-stokes rays).

Deligne also defines a filtration of (E^∇_θ) by the fibers $F_{<\theta_a}$, using the poset structure (using them to filter growth rates)