10/25/2016, Pierre Deligne, Connections with irregular singularities & Stokes structures.

algebraic

X alg. curve $/\mathbb{C}$

V, ∇ (alg.) vec. bdle w/ connection
$\downarrow$
X

$\nabla : V \to \Omega^1_X \otimes V$

satisfying Leibniz

analytic

(analytically, $X = \overline{X} - S$)
cpct Riem. surf — pts.

V, ∇ in analytic sense.
$\downarrow$
X

$\nabla : V \to \Omega^1_X \otimes V$

however, two stories are completely different!

Consider $\boxed{\overline{|d(x,S)}^{-1} =: \overline{\varphi(x)}}$

distance to "∞" w.r.t. a Riemannian metric.

Many alg. questions can be expressed analytically, w/ some growth cond. comparable to φ. Ex:

f polynomial $\longleftrightarrow$ f analytic w/ $|f| \ll \varphi(x)^N$.

A crucial property of such V is they can always be extended:

$\overline{V} / \overline{X}$ (so, can use to give growth conditions)

not unique: if have two extensions differ by "adding poles or zeroes":

$\overline{V}/\overline{X}$, $\overline{V}(-NS) \subset \overline{V}' \subset \overline{V}(-NS)$

$\overline{V}'/\overline{X}$

but this doesn't change "growth rate"

so the equivalence class of such corresponds to

V/X, alg. sections are those w/ polynomial growth $|v| \ll \varphi(x)^N$

analytic

Now, Given V, can look at

$V^{\nabla = 0}$ local sys. of flat sections

In the algebraic category:

In the other way, given local system

$$V \longrightarrow V \otimes \mathcal{O}_X^{an}.$$

Regular singularities:

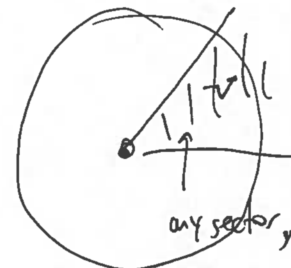
(1) $\exists$ extension $\overline{V}$ over $\overline{X}$ s.t.

$$\nabla : \overline{V} \xrightarrow{\nabla} \overline{V} \otimes \Omega^1(\log S)$$

"only simple poles at ∞".

e.g. $\nabla_{z\partial_z} : \overline{V} \to \overline{V}$.

(1) Analytically



any sector, flat sections satisfy usual growth conditions

$$|v| \ll \varphi(x)^N.$$

$\longleftrightarrow$ these are equivalent!

(2) harder to state, easier to verify computationally, in terms of n^{th} order ODEs, $n = \operatorname{rk} V$.

Given an ODE:

$$(*) \quad \partial^n y + \sum_{1}^{n-1} a_i \partial^{n-i} y = 0$$

gives an asymmetric way of describing V, ∇:

Look at

$(V, \nabla) := \operatorname{jet}_{n-1}(\mathcal{O}), \nabla$

$\exists$ unique ∇ s.t. flat sections are solutions to this ODE.

$$y \longrightarrow y, y', y'', \dots, y^{(n-1)}.$$

In terms of $(*)$ regularity $\Longleftrightarrow$ a_i should have a pole of order $\leq i$.

Why leave regular singularities? Note e^x is a solution to an ODE; & need it e.g., to speak about Fourier transforms. Algebraically, can't make sense of e^x but can make sense of ODE & hence also ODE's satisfied by Fourier transforms of functions.

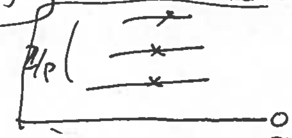
& have analogy of things happening in characteristic p. (exp fcn. has analogue in char. p: $\mathbb{F}_q$ fn. field; look at $\mathbb{F}_q \xrightarrow{\psi} \mu_q$ q^{th} roots of 1 $\longleftrightarrow$ local sys. on line $\mathbb{A}^1$ related to ψ.

cont'd) $\mathbb{A}^1$ not s.c. over $\mathbb{F}_q$: $\mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t^p - t$ Galois covering; w/ $t+1 \mapsto$ same, w/ Galois group $\mathbb{Z}/p$. Now, if have $\mathbb{Z}/p \to \mathbb{Q}_\ell^*$, get ℓ-adic loc. sys.

- there's an analogue of de Rham cohomology (assume $S \neq \emptyset$); coh. of

$$V \xrightarrow{\nabla} \Omega^1 V .$$

"Artin–Schreier sheaf" (very similar to $(V, \nabla) = \mathcal{L}$.) Note e.g.: $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{+} \mathbb{A}^1$, $\mathcal{L} \boxtimes \mathcal{L} = +^* \mathcal{L}$.



p-e… is … π_1 … ex.

Regular singularity case: when considering polynomial resp. 'analytic polynomial' sections, get all the same answer for cohomology.

This will not be true for the irregular case.

Stokes structure.

Have analogous formulae for ℓ-adic sheaves in char. p.

Q: how to find alg. interpretation of de Rham cohomology?

(also useful for things one wants to do w/ Fourier transform?)

Goal: "Classify" those (V, ∇) w/ non-reg. singularities.

really get an equivalence of categories

$$(V, \nabla) \longleftrightarrow \ ?$$ something of analytic ~~…~~ nature.

want:
- compat. w/ pullback
- compat w/ $\otimes$.

} useful for reduction!

useful! e.g., $V \longleftrightarrow GL(n)$-torsor. using pullback, can understand G-torsor for any G.

& e.g., given $z \longrightarrow z^n$ can descend (extract roots w/o losing information)

An example of Euler

(slightly too simple, but many general features are visible).

(first series, tome 14?)

$$\sum (-1)^n n! \qquad \text{Abel summation: look at} \qquad \sum (-1)^n n! z^n = f(z) \quad \& \text{ eval at } z = 1.$$

Observe $f(z)$ satisfies:

$$\boxed{z \partial_z z f(z) + f(z) = 1}$$

~~…~~

The solution of the corresponding homogeneous eq'n " " $= 0$ is $\dfrac{e^{1/z}}{z}$.

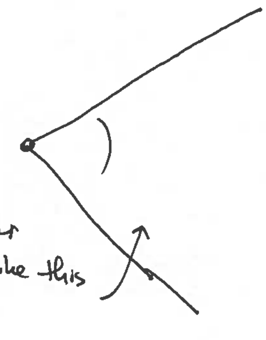
~~The~~ A formula for the solution is:

$$\underbrace{\frac{e^{1/z}}{z}\int_0^z \frac{e^{-1/t}}{t}\,dt}_{(*)} \qquad \left(+ \text{ mult. of hom. solution } \underbrace{\frac{e^{1/z}}{z}}_{(**)}\right).$$

Also, this first order eq'n gives a second order eq'n by taking ∂_z of both sides:

$$\partial_z\left(z\,\partial_z zf + f\right) = 0.$$

In this setting, f $(*)$ gives an ~~as~~ asymptotic expansion ~~sector~~ associated sector of (V,∇) on a sector like this

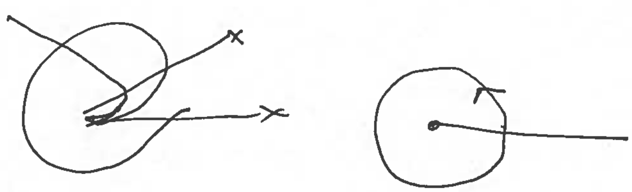


$$f - \sum_1^N \Big(\underbrace{-\!-\!-}_{\text{the series}}\Big) = O(z^{N+1})$$

This is a formal solution of the differential equation, b it diverges. Have also a notion of how fast it ~~grow~~ diverges.

What is the monodromy? Look first at $I = \int_0^z \frac{e^{-1/t}}{t}\,dt$.

changing the path of integration:



start in this direction

After monodromy: $$I_1 = I + \underbrace{\oint e^{-1/t}\frac{dt}{t}}_{2\pi i}$$ integral around 0.

For $(*)$, monodromy:

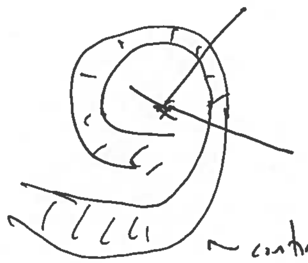
$$f \rightsquigarrow f + 2\pi i\,\frac{e^{1/z}}{z}.$$

(this is another reason the series $\sum(\sim)$ could not have been convergent: no room for this behavior.

In a sector  there is a sol'n which is bounded & no much bigger.

extend the sector:

Cannot distinguish by asymptotic expansion anymore.



~ continue other way

on overlap different function

To distinguish a function, need to look in sectors where it is the smallest solution.

Formal solutions give actual solutions in small enough sectors.

what's happening in general:

First, work formally, over

$\mathbb{C}((z))$. Here, the classification is very simple (up to passing to roots of z)

a) regular connections

b) $\exp\Big(\underbrace{\sum_{1}^{N} a_i z^{-i}}_{P}\Big)$ $\longleftrightarrow$ the vector bundle w/ connection which has this solution: $(\mathcal{O}, d - dP)$.

Thm: Given any $\hat{V}$ on $\mathbb{C}((z))$ $\exists$ a canonical decomposition:

$$\hat{V} = \bigoplus_{P \in \mathcal{P}} (\mathcal{O}, d + dP) \otimes V_P$$

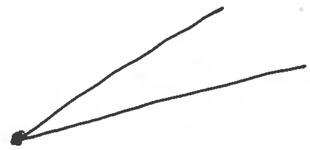
regular.

Given a the vector bundle V, can complete it to $\hat{V}$ & look for formal solutions, using classification

$$(\hat{V}, \nabla) \xleftarrow{\sim} (\ -\ -\ )$$

Given V, ~~$\hat{V}$, in a small sector~~

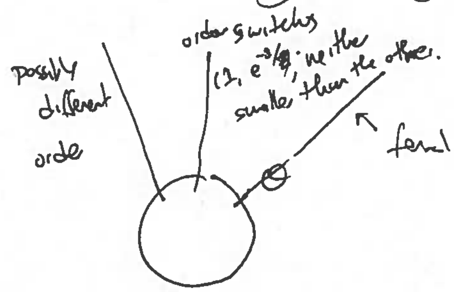
w/ completion $\hat{V}_0 = \hat{V}'$,



on a small sector, solutions of $\hat{V}$ give actual solutions of V.

Work on the punctured disc.

V $\qquad \mathcal{P}$ class of "polar parts" (polynomials) $\quad$ (appears in $\bigoplus_{p\in\mathcal{P}}$ - -)

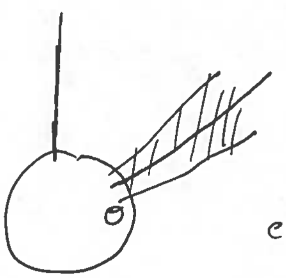


Euler example:

$1 \qquad e^{-1/z}$

to right ↑ much bigger, & to left, much smaller

↑ real blowup of $\mathbb{C}^*$ at 0.



V = local sys. of horiz. sections

~~some dates~~

each section $v \longrightarrow$ growth rate $p \in \mathcal{P}$.

For a general angle, this gives a filtration by growth rates.

Get a filtration $V^{\theta}_{<p}$:

At a ray of crossing,

(or any point), $\exists$ a non-unique decomposition $V = \bigoplus V_p$

- on a general ray, unique up to "lower triangular." (unipotent subgroup given by ordering.)

- At a special ray, unique up to fewer possibilities; intersection of possibilities to the right & the left

"Stokes line/ray" →

Thm: There is an equivalence of categories:

$$(V, \nabla) \longleftrightarrow (V, Fil)$$

filtration in every direction.

For these objects, $\otimes$ & pullback are relatively clear; & is true only after extracting some nth roots, or allowing $\mathcal{P}$ to be not a set but a finite local system on S^1 w/ some ordering.

To take duals: Replace $\mathcal{P}$ by $-\mathcal{P}$, & given a filtration on dual, get a filtration (up to relabeling)

tensor product: Give $\mathcal{P}$, Q, look at all $\mathcal{P}+Q$,

$$\& \quad V, W \longmapsto V \otimes W = \bigoplus V_{\mathcal{P}} \otimes W_Q \text{ formally}$$

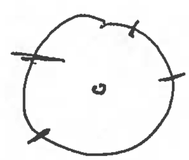
Ramification: replace z by $z^{1/t}$ & pull back.

Rmk: Since there is a cat. w/ $\otimes$, it's the analogue of a group $\rightsquigarrow$ analogue of "local π_1".

De Rham cohomology in this language.

Have: X, V + meromorphic data

V



Thm: $H_{dR}(V, \nabla) \cong H(\tilde{X}, V^{\leq 0})$

the real blow-up of X at 0:

Have sheaf: $V^{\leq 0}$ (jumps at Stokes rays) constructible; (sections with $\exp(\text{—})$ bounded positively)

cpct. support.

$$\& \quad H_{dR_c}{}^{!} \cong H(\tilde{X}, V^{<0})$$

There's another description, using duality between homology & cohomology.

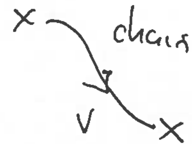
Have

$\mathcal{V} \rightsquigarrow V$ local sys. & $\mathcal{V}^\vee \rightsquigarrow V^\vee$.

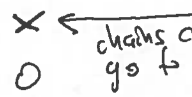
bundle w/ connection

Describe chains w/ coeffs. of $V^\vee$:

outside 0:

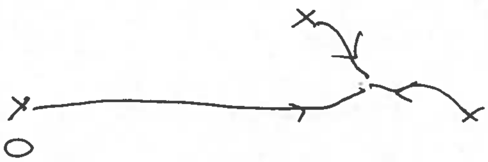


chain equipped w/ section $v \in V^\vee$.

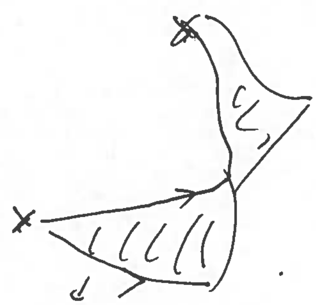


chains can go to missing point as long as the section decreases fast.

To get a cycle: sum should be zero:



& homology:



$v^\vee < 0$.

gives $H_B(V^\vee)$ Betti coh. of local sys. w/ stokes structure.

Have a natural pairing

$$H_{dR}(\mathcal{V}) \otimes H_B(V^\vee) \xrightarrow{\int} \mathbb{K}$$

$f \cdot \omega$ just by integrating: converges b/c of decay

section, diff form

Thm: Gives a perfect pairing.

Suppose have V vec. bdle w/ connection. Can take

$$\underline{H^1_{dR}(V \otimes e^{\lambda x}), \quad \nabla_{GM}}$$

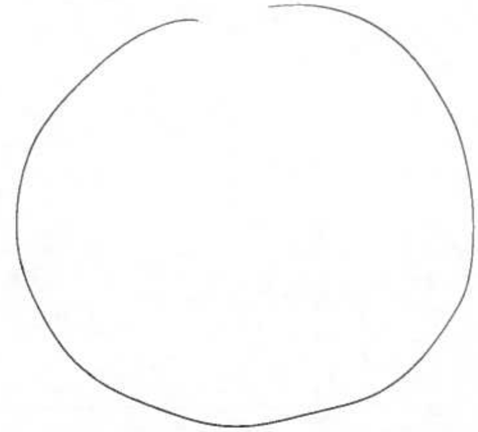
set ↙ outside exceptional vals

View as a family over (x, λ) two parameters

Guess: a way of understanding is in terms of circles what happens as λ changes,
e.g. also the <u>Fourier transform</u> of vector bundle w/ connection:

~~To~~ recall

(exchange $x \leftrightarrow \partial_x$).



In the equivalence of categories,

given (V, F), it makes sense to speak of
↑ filtration

$Gr_F(V)$; its indexed
↑ even on Stokes ray, have "direct sum decomp w/ ambiguity"

by $\mathcal{P}$. For each $\mathcal{P}$,

$Gr_{\mathcal{P}}(V)$ gives local system on the circle. This gives the <u>completion functor</u>.

~~Rec~~

(Recall formula $\hat{V} = \oplus (\exp \mathcal{P}) \otimes \text{reg.}$
alg: group = monodromy found + things of rank 1),
In char. p, no distinction between <u>completion</u> & Henselization)

complete, analytic, meromorphic condition
↑ $\oplus(\exp P) \otimes \text{reg.}$ ↑ monodromy ↑ Stokes story

<u>Rmk</u>: note ~~...~~ the monodromy is <u>killed</u> or <u>created</u> (just different) in $Gr_{\mathcal{P}}(V)$ relative the analytic setting.

Applications: what's the biggest sector can find solution w/ some asymptotic section?

· ~~[illegible]~~ (e.g. Euler ex: sector needs to be big enough to contain some real positive part)

· general rules about angles one can take to find solutions in sectors.

(unitary analogue of this stuff: look at work of Takeo Mochizuki).