11/1/2016, L. Borisov, <u>the Dubrovin connection & a first statement of the Gamma conjectures</u>

F := Fano manifold

<u>Reference</u>: [Galkin – Golyshev – Iritani].

$$H^{\text{even}} = \bigoplus_{p \geq 0} H^{2p}(F, \mathbb{C}).$$

There is a family of products on H^{even}, defined by "<u>big quantum cohomology</u>" (restr. to <u>even</u> part)

$$\langle \alpha_1 *_\tau \alpha_2, \alpha_3 \rangle = \sum_{d \in \mathrm{Eff}(F)} \sum_{n \geq 0} \frac{1}{n!} \int_{[\mathcal{M}_{0,3+n,d}]_{\text{virt}}} ev_1^* \alpha_1 \wedge ev_2^* \alpha_2 \wedge ev_3^* \alpha_3 \wedge \prod_{i=4}^{n+3} ev_i^* \tau$$

(where $\alpha_i, \tau \in H^{\text{even}}$).

<u>Magic</u>: For any τ, $- *_\tau -$ is commutative (obvious) and associative (non-obvious!)
(associativity $\Leftarrow$ relation in $\mathcal{M}_{0,4}$)

<u>Q</u>: <u>Convergence</u>?

- For $\tau = 0$, we're just working with $\mathcal{M}_{0,3,d}$ & $\dim[\mathcal{M}_{0,3,d}] = c_1(F) \cdot d + \dim F + \underset{(= \#\text{marked pts.} - 3)}{\overset{\shortparallel}{0}}$.
  so for sufficiently large d, this count will be zero, & count is ~~polynomial~~ has only finitely many terms; so convergence is a non-issue.
- For general τ, just assume convergence at some point (otherwise, we'd have to introduce <u>Novikov</u> coefficients, ...).

<u>Rmk</u>: Convergence seems to be helpful to formulate "Gamma Conjecture II.." (but not necessarily I ?) & ~~exists~~ holds in the various examples. — work formally

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Consider the trivial bundle

$$H^{\text{even}} \times (H^{\text{even}} \times \mathbb{CP}^1) \longrightarrow (H^{\text{even}} \times \mathbb{CP}^1).$$

There's a <u>connection</u> (meromorphic) (non-trivial!) on this bundle, defined by, at a point (τ, z) on the base,

$\alpha \in H^{\text{even}}$, $\nabla_{\partial_\alpha} = \frac{\partial}{\partial \alpha} + \frac{1}{z}(\alpha *_\tau -)$ (flatness of ∇ in ∂_α directions $\Longleftrightarrow$ associativity of $*_\tau$

2) $$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}\left(E *_\tau -\right) + \mu.$$

where • $\mu : H^{even} \to H^{even}$ is a grading operator

$$\mu\Big|_{H^{2p}} = \left(p - \tfrac{1}{2}\dim F\right) Id$$ (this setup satisfies $\langle \mu\alpha, \beta\rangle = -\langle \alpha, \mu\beta\rangle$)

• $$E = c_1(F) + \underbrace{\left(1 - \tfrac{1}{2}\dim F - \mu\right)}(\tau).$$

"$1 - \frac{\deg \tau}{2}$"; when, on small QH, $\deg \tau = 2$, this term vanishes.

How to see

Flatness:

• $[\nabla_{\partial_\alpha}, \nabla_{\partial_\beta}] = 0 \Leftrightarrow$ associativity

• $[\nabla_{\partial_\alpha}, \nabla_{z\partial_z}] = 0 \Longleftrightarrow \dim [M_{0,n,d}]^{virt}$ this term encodes dimensions of moduli spaces.

$$\| \\ c_1(F)\cdot d + \dim F + (n-3).$$

(there's an interpretation of connection in z-direction from $\mathbb{C}^*$-equivariant story; u connections, etc.) also, it's "what remains of the grading."

Mostly interested in $\tau = 0$, in which case the connection is:

$$\nabla_{z\frac{d}{dz}} = z\frac{d}{dz} - \frac{1}{z}\left(c_1(F) *_0 -\right) + \mu.$$

[reference: connection in the z direction seems to first appear in Dubrovin's works].

We're interested in $H(z)$ values in H^{even} with the property that

$$\left(z\frac{d}{dz} - \frac{1}{z}\left(c_1(F)*_0 -\right) + \mu\right) H(z) = 0.$$ flat sections

or more generally $H(\tau, z)$ satisfying

$$\nabla_{\partial_\alpha} H(\tau, z) = 0$$

$$\nabla_{z\partial_z} H(\tau, z) = 0.$$

Define $S(\tau, z)$ via
$\in$
$\mathrm{End}(H^{even})$. (this is what fund. solutions are; hit w/ constant vector to get actual solution)

$$\langle S(\tau,z)\beta_1, \beta_2\rangle := \langle \beta_1, \beta_2\rangle + \sum_{\substack{n\geq 0,\, d\in \mathrm{Eff} \\ (n,d)\neq(0,0)}} \frac{1}{n!}\int_{[\mathcal{M}_{0,n+2,d}]_{virt}} \frac{ev_1^*\beta_1}{-z-\psi_1}\cdot ev_2^*\beta_2 \cdot \prod_{i=3}^{n+2} ev_i^*\tau.$$

($\beta_i \in H^{even}$; S is $\mathrm{End}(H^{even})$-valued)

(this looks good/single-valued/etc. near $z\approx\infty$).

think of this via expanding as a power series; (formal) expansion in $\frac{1}{z}$ (ψ_1 nilpotent, and assuming convergence, could do for any z !).

where $\psi_1 = c_1(L)$ ← universal cotangent bdle at first marking

In fact, it turns out that

$\nabla_{\partial_\alpha} S(\tau, z) = 0$; this follows from topological recursion (expressing ψ_1 as a particular boundary divisor, recursively expanding — !)

Unfortunately,

$\nabla_{z\partial_z} S \neq 0$; but can remedy this by multiplying by something which is τ-independent. (w/o affecting)

Fix: Define $\rho = (c_1(F)\cdot -) \in \mathrm{End}(H^{even})$, and look at

← std product by c_1, not quantum.

$$S(\tau,z)\, z^{-\mu} z^{\rho}$$

$z^\rho = e^{\rho \log z}$ ← poly. in log z; ρ is nilpotent. (fixing monodromy).

this means $e^{-\mu \log z}$ ← picking "usual branch" of log, to resolve monodromy at ∞; taking "cut."

(or directly defined, b/c μ is a bunch of diagonal matrices; has a bunch of sq. roots of z; need to again chose a branch).

(can get rid of $\sqrt{\ }$/branch by using $2c_1(F)$.

Rmk: So far, we're near $z=\infty$? so haven't run into Stokes data.

Galkin-Iritani: can rescale by degrees, identify $z\mapsto z^{-1}$, etc. & in Fano case, this identifies $\nabla_{z\partial_z}$ w/ ∇_{∂_α}.

(Rmk: $z^{-\mu}$ makes "$+\mu$" in connection disappear; so at ∞, next term is $\frac{1}{z}(c_1(F)*_0 -)$; if rescale deg. $2p$ part by z^p; highest pole part will be $c_1(F) *_{\dots} (-)$! ...).

Claim: now, $\nabla_{z\partial_z} S(\tau, z) z^{-\mu} z^{\rho} = 0$ now.

Some properties (at $\tau = 0$) now: (which would need to be verified).

(a) $\nabla_{z\frac{d}{dz}} \overbrace{S(0, z) z^{-\mu} z^{\rho}}^{\text{"Solution"}} = 0$, and

(b). $\lim_{z\to\infty} z^{\mu} S(0,z) z^{-\mu} = Id$ (formal limit, at least) ($\Longleftrightarrow \lim_{z\to\infty} z^{-\mu}$"Solution"$z^{\rho} = Id$). (Rmk: this seems to use <u>Fano</u>).

<u>Prop</u>: These determine $S(0,z)$ uniquely. (***)

<u>Pf</u>: If S_1 another ~~...~~ satisfying (a), (b), then

$$S_1(z) z^{-\mu} z^{\rho} = S(0,z) z^{-\mu} z^{\rho} C \quad \in End(H^{even})$$

$$z^{\mu} S_1 z^{-\mu} = z^{\mu} S(0,z) z^{-\mu} \, z^{\rho} C z^{-\rho}$$

but z^{ρ}, $z^{-\rho}$ $= Id +$ poly$(\log z)$, $= Id +$ poly$(\log z)$.

$z^{\mu} S_1 z^{-\mu} \xrightarrow{z\to\infty \text{ by (b)}} Id$; $z^{\mu} S(0,z) z^{-\mu} \xrightarrow{z\to\infty \text{ by (b)}} Id$; $\Rightarrow$ as $z\to\infty$, $z^{\rho} C z^{-\rho}$ goes to Id.

so

$$z^{\rho} C z^{-\rho} = C + \sum_{k>0} c_k z^k \to Id.$$

means C itself is identity.

$\Rightarrow C = Id$.

Consider the case $F = \mathbb{P}^{N-1}$,

~~write~~ look at $\left(z\frac{d}{dz} - \frac{1}{z}(c_1(F) *_0 -) + \mu\right) H(z) = 0$. & look at solutions as

$\infty \leftarrow$ $z \to 0^+$ along $\mathbb{R}_+$.

Up to scaling, there is <u>one solution</u> f with <u>smallest</u> asymptotics at $z\to 0^+$, $f(z) \sim e^{-N/z} O(z^{-m})$ ($N = 1 + \dim \mathbb{P}^{N-1}$).

(there are also formally $e^{-N(\text{root of } 1)/z} O(-)$ ... but these are bigger along $\mathbb{R}_+$),

"$e^{-N\chi/z}$" (χ N^{th} root of 1.) are ~~...~~ minimal in other sectors.)

Claim: (Gamma Conjecture I for $\mathbb{CP}^{N-1}$).

Up to constant, this smallest solution

$$f(z) = \underbrace{S(0,z)z^{-\mu}z^{\rho}}_{\text{End-valued}}\ \underbrace{\Gamma(\mathbb{CP}^{N-1})}_{\in H^{even}} \leftarrow \text{gamma class of } \mathbb{CP}^{N-1}$$

(think of Γ as End valued, (cup), apply to 1)

Let's check this.

- $H^{even} = \bigoplus_{i=0}^{N-1} h^i$, $h :=$ hyperplane class, and

$$h *_0 h^i = \begin{cases} h^{i+1}, & i \le N-2 \\ 1, & i = N-1. \end{cases}$$

to see this, note

$$\langle h *_0 h^i, h^j \rangle = \sum_d \int_{\overline{M}_{0,3,d}(\mathbb{CP}^{N-1})} ev_1^* h \, ev_2^* h^i \, ev_3^* h^j.$$

$\dim = N \cdot d + N - 1$.

contributions when:

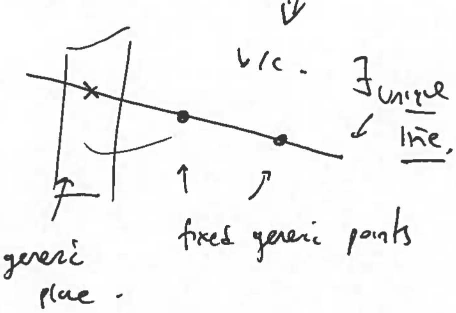
$$1 + \underset{\underset{N-1}{\wedge\!\!|}}{i} + \underset{\underset{N-1}{\wedge\!\!|}}{j} = Nd + N - 1.$$

$d \geq 2$: nothing

$d = 0$: usual product.

$d = 1$: $i = j = N-1$, & integral is:

$$\int_{Gr(\text{lines in } \mathbb{CP}^{N-1}) \,=\, \overline{M}_{0,3,1}(\mathbb{CP}^{N-1})} \_\_\_ = 1$$



Eigenvalues of $c_1(F) *_0 -$:

$= N\xi$ where $\xi^N = 1$ (N^{th} root of 1).

Note:

$$N = \max_{\substack{\lambda \text{ eigenvalue} \\ \text{of } c_1(F) *_0 -}} |\lambda|.$$

(more generally, the max eigenvalue is something non-obvious...)

(Rmk: note the max eigenvalue determines the smallest asymptotics as $z \to 0^+$ along $\mathbb{R}_+$, in more general formulation of Gamma conjecture I).

Write:

$$\underset{\in H^{even}}{f(z)} = \sum_{i=0}^{N-1} f_i(z) z^{i - \frac{(N-1)}{2}} h^{N-1-i}.$$

Have eq'n: $\nabla_{z\,d/dz} f = 0$

using:

$$z \frac{d}{dz} f_i(z) = \begin{cases} N f_{i+1}(z) & i = 0 \ldots N \\ N z^{-N} f_0(z), & i = N-1, \end{cases}$$

rewrite ~~solution of~~ equation on f as an Nth order ODE in f_0:

$$\left(\left(z \frac{d}{dz} \right)^N - N^N z^{-N} \right) f_0(z) = 0 . \qquad (*)$$

can write as a formal power series & solve for fundamental solution!

(formal, at ∞).

Fundamental solution: (f_0, f_0),

$$\underset{\substack{\Uparrow \\ (H^{even})^\vee\text{-valued}}}{\Pi(z)}(\overset{H^{even}}{\alpha}) = \int_{\mathbb{CP}^{N-1}} \sum_{n=0}^{\infty} \alpha \frac{z^{-N(n-h)}}{((h-n)(h-n+1)\cdots(h-1))^N} \qquad (+)$$

verify it satisfies $(*)$.

Then calculate all other f_i's by $z\frac{d}{dz}$ derivatives;

plug in; to get Φ.

From here, write the (H^{even}) endomorphism valued function

$$\Phi(z)(\alpha) = \sum_{i=0}^{N-1} z^{i-(N-1)/2} h^{N-1-i} \int_{\mathbb{CP}^{N-1}} \sum_{n=0}^{\infty} \frac{(h-n)^i \alpha z^{-N(n-h)}}{((h-n)(h-n+1)\cdots(h-1))^N}$$

claim:

$\phi(z)$ is $S(0,z) z^{-\mu} z^{\rho}$. (by Calculation, check it satisfies (c) & (b) in $(***)$!)

(this is all formal, at ∞ so far. Now need to go near 0, & look at leading order solution).

Now, write

$$\psi(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)^N z^{Ns} \, ds.$$

$\ast \ -2 \quad \ast \ -1 \quad \ast \ 0$

(want c positive real part; then convergence essentially by Stirling's formula;

(avoid poles of Γ) & doesn't matter which c you use (under conditions on z)

(actually angle of z should be in certain sector, between $-\pi/2$ & $\pi/2$?)

check: (1) ψ satisfies the differential equation:

(explicit verification: taking derivatives, rewrite derivative as $\int$ for different c, use independence of c.)

It is checkable that

$$(2) \quad \Psi(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)^N z^{Ns}\, ds \underset{\substack{z\to 0^+ \\ \text{along } \mathbb{R}_+}}{\sim} C\, z^{(N-1)/2}\, e^{-N/z}\,(1+O(z)).$$

small asymptotic behavior.

[c.f. reference to Braaksma's work which may explain this clearly; original references go back to 1946 & the 1906!!]

use $\Gamma(s)^N \sim \Gamma(Ns - \text{const.})$ up to terms which need to be checked.

Now, to prove Gamma conjecture I:

(applied to 1, to get a functional),

Need to argue $\Psi(z) = \Phi(z)\,\Gamma(\mathbb{CP}^{N-1})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1+h)^N$.

(This is just for f_0; but if have it for f_0, have it for all others by taking partial derivatives)

$=$ sum $\sum$ residues, (by estimating as push c to left of poles).

& calculate residues work out explicitly to be $(\dagger)$.

Let's compute one:

$$\operatorname{Res}_{s=-n} \Gamma(s)^N z^{Ns}$$

$$= \operatorname{Res}_{s=0} \Gamma(s-n)^N z^{N(s-n)} = \operatorname{Res}_{s=0} \frac{\Gamma(1+s)^N z^{N(s-N)}}{s^N (s-1)^N \cdots (s-n)^N}$$

s^N ← term blowing up.

$$= \text{coeff. of } s^{N-1} \left(\frac{\Gamma(1+s)^N z^{N(s-N)}}{(s-1)^N \cdots (s-n)^N} \right)$$

$$= \int_{\mathbb{CP}^{N-1}} \left. \underbrace{\Gamma}_{\Gamma \text{ class.}} \frac{(1+h)^N z^{N(h-n)}}{(h-1) \cdots (h-n)^N} \right\} \Pi(z).$$

(this is a special case of other Gamma conjectures; replace $1 = ch(\mathcal{O})$ by ch(line bundles), & study approaches along other lines).