11/4/2016, Hiroshi Iritani; A proof of Gamma conjecture in some cases via mirror symmetry

Γ-conjecture: compatibility between the Stokes structure of quantum differential equation and the $\hat{\Gamma}$-integral structure.

[KKP, GGI].

Start with the B-model (LG model).

Consider $f: Y \longrightarrow \mathbb{C}$ $\quad Y$ some affine variety, in this case $Y = (\mathbb{C}^\times)^n$.

(assume: • f has only isolated critical points

• f is "tame" (there are many notions of "tame"; e.g., assume $|\partial f| > \varepsilon$ outside of a cpct subset for some $\varepsilon > 0$)

(allows one to e.g., do Morse theory for $Re(f)$)).

$\rightsquigarrow TEP(f)$ (using notation of [Hertling],) connection on $\mathbb{C}_z$ ← sometimes denoted by 'u' [KKP]

original references [K&M. Saito, Sabbah, Douai-Sabbah] ← closest to current discussion

Define $\mu_x = \dim_{\mathbb{C}} \underbrace{\mathcal{O}_{Y,x}/(\partial f)}$, $x \in Y$ crit. point,

(Milnor #; Jacobi ring).

and $N = \sum_{x \in Crit(f)} \mu_x$. $\quad \mathbb{Z}^N$

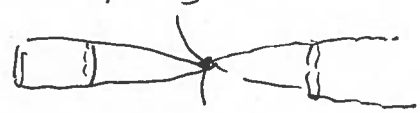
$\quad\quad\quad\quad \cong$

Local system over $\mathbb{C}^*_z$ $\quad R^\vee_z := H_n(Y, \{Re(f(y)/z) << 0\}; \mathbb{Z})$

"the space of Lefschetz thimbles"

There's a natural pairing

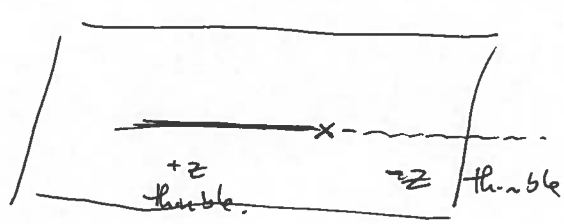
$$I: R^\vee_{-z} \times R^\vee_z \longrightarrow \mathbb{Z} \quad \text{perfect pairing}$$



(perfect pairing), ~~and~~ Writing

$$R_z = Hom(R^\vee_z, \mathbb{Z})$$

$$= H^n(Y, \{Re(f/z) << 0\}),$$



there's an induced pairing

$$I^\vee: R_{-z} \times R_z \longrightarrow \mathbb{Z}.$$

(want to extend loc. sys. over $\mathbb{C}^*_z$ to meromorphic connection)

Oscillatory integral: Given $\omega \in \Omega^n_Y[z]$, consider $\left[e^{f/z}\omega\right] \in R \otimes \mathcal{O}^{(an)}_{\mathbb{C}^*}$ section.

depends on z polynomially

analytic.

$\mathcal{R}$ (v.b. assoc. to lattice R).

Lefschetz thimble.

$$\left[e^{f/z}\omega\right] : [\Gamma] \longmapsto \int_\Gamma e^{f/z}\omega \in \mathbb{C}.$$

$[\Gamma] \in R^\vee_z$

$\rightsquigarrow$ extension of $\mathcal{R}$ across $z=0$.

(Rmk: everything in analytic topology)

define $\mathcal{R}^{(0)} \subset i_*\mathcal{R}$ to be the submodule generated by the oscillatory integrals $\left[e^{f/z}\omega\right]$,

$i : \mathbb{C}^* \to \mathbb{C}$

locally free $\mathcal{O}_{\mathbb{C}}$-module, of rank N.

Also need a pairing:

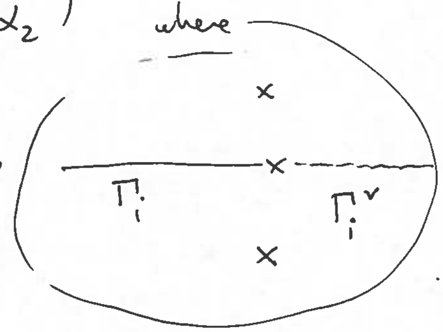
$$P : (-)^*\mathcal{R} \otimes \mathcal{R} \longrightarrow \mathcal{O}_{\mathbb{C}}. \quad \text{where } (-) : \mathbb{C} \to \mathbb{C}, \; z \mapsto -z$$

$$(\alpha_1, \alpha_2) \longmapsto \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i z)^n} I^\vee(\alpha_1, \alpha_2)$$

($n = \dim Y$, $N = \sum \mu_x$).

(non-trivial thing: P is regular at $z=0$, and also non-degenerate there)

Can compute
$$P(\alpha_1, \alpha_2) = \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i z)^n} \sum_i \left(\int_{\Gamma_i^\vee} \alpha_1\right)\left(\int_{\Gamma_i} \alpha_2\right)$$
where



(always has order n 0 at $z=0$, so this cancels).

Also, have a connection:

$$\nabla^B = \nabla^{GM} - \frac{n}{2}\frac{dz}{z}$$

these shifts come from the fact that we're mapping to $\mathcal{O}_{\mathbb{C}}$ rather than $z^n\mathcal{O}_{\mathbb{C}}$; can set up theory w/ latter & eliminate shifts.

Say $\left(\mathcal{R}^{(0)}, \nabla^{B}, P\right) =: \underbrace{\mathrm{TEP}}_{\mathbb{Z} \text{ (b/c also have } \mathbb{Z} \text{ structure)}}(f) \quad + \quad R : \mathbb{Z}\text{-structure}$

Remarks:

- Note that

$$R^{(0)}\Big|_{z=0} \cong \mathrm{Jac}(f)\cdot\omega_0 \quad \text{free module over Jacobian ring.}$$

& $P|_{z=0}$ = residue pairing.

- Can set up theory w/ parameters; if have

$$\underset{\text{parameter space}}{F : Y \times \underbrace{\mathcal{M}}} \longrightarrow \mathbb{C} \quad \rightsquigarrow \mathrm{TEP}(F) : \text{connection on } \underbrace{\mathcal{M}} \times \mathbb{C}_z$$

$\updownarrow$ Kähler moduli space on A-side.

- ∇^{GM} has an explicit description:

$$\nabla^{GM}_{\frac{\partial}{\partial z}}\left[e^{f/z}\omega\right] = \left[\left(\frac{\partial \omega}{\partial z} - \frac{f}{z^2}\omega\right)e^{f/z}\right]$$

pole of order 2 at $z=0$; connection could be irregular.

(critical values of f will somehow be responsible for the irregular singularity)

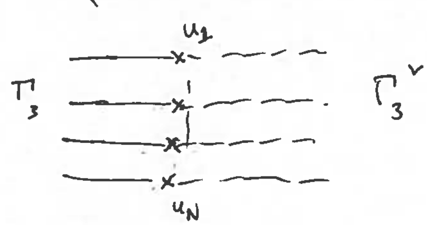
why? (highest pole order part at $z=0$; forgetting about z^2, this is $f \times \mathrm{Jac}$, which picks out critical values) (using $R^{(0)}_{z=0} = \mathrm{Jac}$).

Remark: give a trivialization of $\mathrm{TEP}(f)$: existence of primitive form.
(a big problem in singularity theory)
- a priori a non-trivialized vector bundle

(under mirror symmetry, get one such, but could have many trivializations)

Formal structure of $\mathrm{TEP}(f)$: Assume, for simplicity that crit. points are all non-degenerate, w/ pairwise distinct critical values. ($\leftrightarrow QH^*$ mirror is semisimple)

write critical values $u_1, \rightarrow u_N$
cr_i: i^{th} crit. point
$f(cr_i) = u_i$.



Have the following asymptotic expansion:

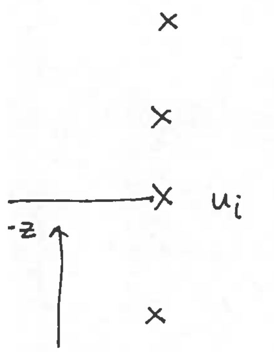
Let $\omega = \phi(y) \underbrace{dy_1 \cdots dy_n}_{\text{local coordinates near } i^{th} \text{ critical point, } cr_i}$

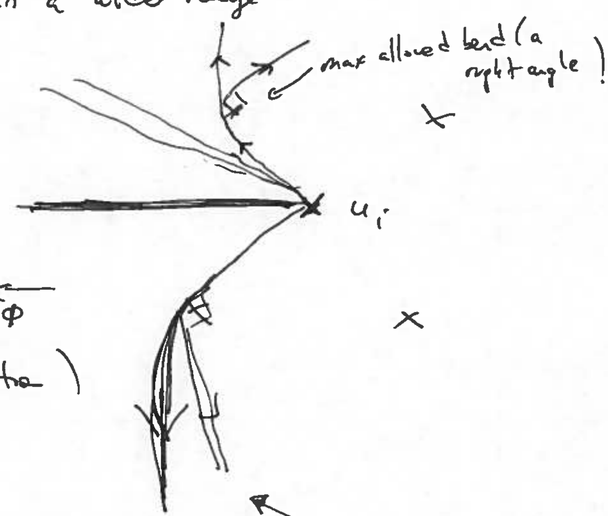
Then,

$$\int_{\Gamma_i} e^{f/z}\,\omega \underset{z \to 0}{\sim} \frac{e^{u_i/z}}{(-2\pi i z)^{n/2}} \left(\frac{\phi(cr_i)}{\sqrt{\mathrm{Hess}(f)(cr_i)}} + O(z) \right)$$

(do by approximating $\int$ as a Gaussian integral near the critical point!)

Actually the expansion is correct in a wider range





Γ_i in direction $-z$. (we fix phase z; & Γ_i in opposite direction) required to make asymptotic expansion correct.

This expansion holds wherever we can define (continuously) (z, and) Γ_i, so that $\mathrm{Re}(f/z)$ decreases monotonically along Γ_i; the max allowed bend (z and Γ_i with it) is then right angles here of Γ_i which gives correct expansion,

In particular, the size of this sector is always $> \pi$.

<u>Hukuhara – Turrittin–Levelt thm</u>; (in this context) [c.f. Hertling – Sevenheck reference]

In this setting,

The theorem states:

① $TEP(f)_0 \otimes_{\mathbb{C}\{z\}} \mathbb{C}[[z]] \cong \bigoplus_{j=1}^{N} \left(\mathbb{C}[[z]],\ d + d(u_j/z),\ \text{std. pairing} \right)$

w/o these terms, trivial.

(formal ~~of~~ germ at 0).

$\in$

$$[e^{f/z}\omega] \longmapsto \left(\text{asymp. expansion of } (-2\pi z)^{n/2} e^{-u_j/z} \int_{\Gamma_j} e^{f/z}\omega \right).$$

(this map intertwines P w/ standard pairing;)

(Recall $P(\alpha_1, \alpha_2) = \sum \int_{\Gamma_j^\vee} \alpha_1 \int_{\Gamma_j} \alpha_2$; asymptotic expansion intertwines this w/ std. pairing)

Rmk: at $z=0$, this isomorphism is the decomposition of the Jacobian ring into factors of $\mathbb{C}$.

② The isomorphism in ① can be lifted to a unique isomorphism over a sector $(\phi - \pi/2 - \varepsilon,\ \phi + \pi/2 + \varepsilon)$;

relies on getting a sector this large

$$TEP(f)_0 \otimes \mathcal{A}_S \xrightarrow{\cong} \bigoplus_{j=1}^{N} \left(\mathcal{A}_S,\ d + d(u_j/z) \right)$$

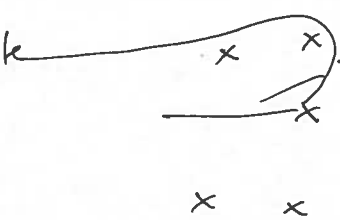
a sector

$\mathcal{A}_S :=$ hol. fns. on S admitting asymp. expansion at $z=0$ along the sector.

(this gives the Stokes structure)

(this result is part of the general theory of connections w/ this formal type.)

(why need a sector for uniqueness? note

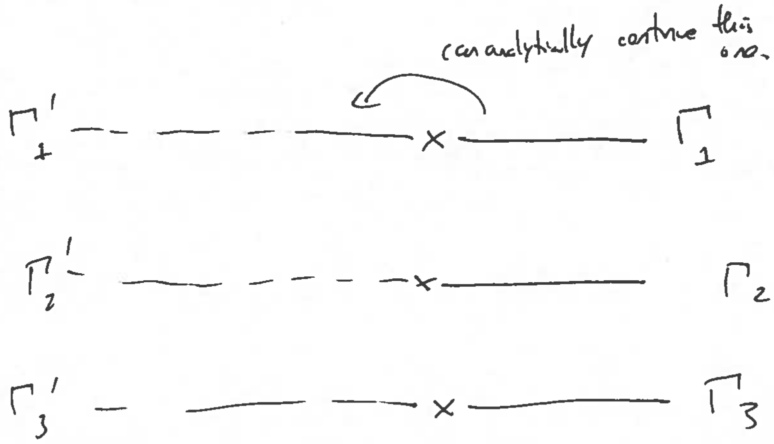


both arcs have same asymp. connection over $z=0$, but if want unique isomorphism in a sector, only one is correct.

This isomorphism will be different for different ϕ:

To compare isomorphisms:

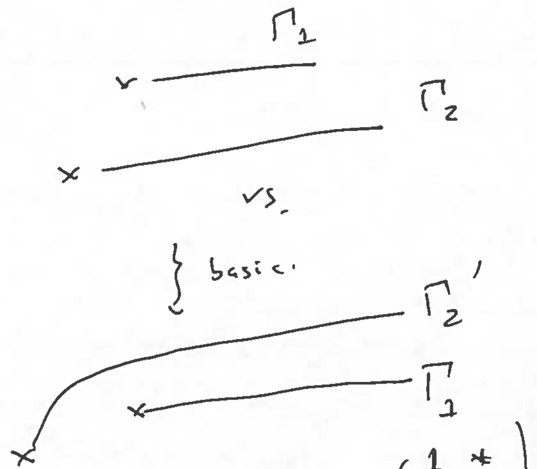
(Assumptions: no two critical values differ by any multiple of ϕ: want $e^{i\phi} \neq u_i - u_j$.
↑ if this happens, bad direction / Stokes rays)



Stokes matrix $S_+ = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$ upper triangle.

S_-

or:



Stokes factor $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

If start w/ general ~~flat~~ connection, $*$ may not be $\in \mathbb{Z}$.
In this case, geometry dictates $* \in \mathbb{Z}$;
can be computed as intersection of vanishing cycles.

$$S_{ij} = \#\left(e^{-\pi\sqrt{-1}} \Gamma_i \cap \Gamma_j \right)$$ (Picard-Lefschetz type intersection.

↑ could even just rotate by small #

On the A-side, have a similar structure: (this setup is for general X).

$$TEP^A(X) = \left(\begin{matrix} H \times (H \times \mathbb{C}_z) \\ \downarrow \\ H \times \mathbb{C}_z \end{matrix}, \nabla^{Dub}, P \right) + \hat{\Gamma} \text{ integral structure}$$

↑ Poincaré pairing

Restricting to Fano case: • know convergence

in general, • $z \to \infty$ limit $\Longleftrightarrow$ Euler flow (or inverse flow) on H.

• In Fano class, $z \to \infty \Longleftrightarrow$ large radius limit (b/c Euler = c_1 flow, & is ample).

To define $\hat{\Gamma}$ structure; just on H^{even} for now:

Consider

$$K^0_{top}(X) \longrightarrow \left\{ \begin{array}{l} \text{multi-valued} \\ \text{flat sections of } \nabla^{Dub} \end{array} \right\}$$

asymptotically (Fano case) as $z \to \infty$ (or large radius limit in general case).

$$V \longmapsto S_V(\tau, z) \sim \underbrace{e^{-\tau/z}}_{\frac{1}{(2\pi)^{n/2}}} z^{-\mu} z^{c_1(X)} \left(\hat{\Gamma}(TX) \cup (2\pi i)^{\frac{\deg}{2}} ch(V) \right)$$

$(\tau, z) \in H \times \mathbb{C}^*$

as $\tau \to LRL$. . "leading term of J-function"

large radius limit.

(if write $\tau = \tau^{(2)} + \tau'$ & send $\mathrm{Re}\left(\int_d \tau^{(2)}\right) \to -\infty$ for any effective curve class
H^2 class

$\tau' \to 0$.

LRL appears, b/c if consider. big QH^*:

have $\sum N_d e^{\int_d \tau^{(2)}}$ want limit in which all these die except $d=0$.

Rmk: The Dubrovin also has regular singularity as $\tau \to LRL$, so can't specify value, just asymptotic form.

In Fano case, can also specify $z \to \infty$ asymptotic form.

1) (b/c we send everything to a flat section, obvious that preserved by ∇^{Dub})

2) monodromy invariance is easy to check.

Mirror symmetry statement: this TEP structure w/ $\hat{\Gamma}$ str. $\Longleftrightarrow$ TEP(f) + R.

Rmk: If X and f are mirror to each other, want

$u_1, \ldots, u_N$ crit. values of f $\longleftrightarrow$ eigenvalues of $E *_\tau -$.

Dubrovin conjecture: Restrict to X Fano (though Gamma II may make sense outside of this).

(1) $QH^*(X)$ is semisimple $\Longleftrightarrow$ $D^b(X)$ has a full exceptional collection. $\{E_i\}$

(2) When (1) holds, the Stokes matrix $S_{ij} = \chi(E_i, E_j)$
Stokes of TEP$^A(X)$.

(3) Central connection matrix. (will omit, b/c $\Gamma\mathrm{II}$ should refine (3)).

$\Gamma\mathrm{II}$ (GGI): For $\tau \in H^*(X)$ and an admissible ϕ $(e^{i\phi} \nparallel u_i - u_j)$, not parallel

when (1) holds,

$\exists\ \{E_i\}$: full exceptional collection (may depend on τ, ϕ.) such that

$S_{E_i}(\tau, z)$ defines an analytic lift of the formal decomposition of $TEP^A(X)_\tau$

at $z=0$ over the sector $(\phi - \frac{\pi}{2} - \varepsilon,\ \phi + \frac{\pi}{2} + \varepsilon)$.

(If one knows $\Gamma\mathrm{II}$, then Dubrovin (2) follows; from relationship of $\chi(E_i, E_j)$ to Γ/ sqrt. of Todd).

Proof for $\mathbb{P}^n$ $(n=2)$

We know: $TEP^B(f) \cong TEP^A(\mathbb{P}^n)$.

Mirror $f = x + y + \dfrac{q}{xy}$.

(identifying something (primitive form) $\longleftrightarrow 1$)

$3\omega q^{1/3}$ ×

$3q^{1/3}$ × $\longleftarrow$ $\Gamma_{\mathbb{R}} = \{(x,y) \mid x>0,\ y>0\}$. $\xleftrightarrow{\text{want}}$ $\mathcal{O}_{\mathbb{P}^n}$.

$3\omega^2 q^{1/3}$ ×

$\underbrace{(1, S_0(\tau, z))}$

So, want to show

$$\boxed{\int_{\Gamma_{\mathbb{R}}} e^{-f/z} \frac{dx}{x} \frac{dy}{y} = (2\pi z)^{n/2} \int_{\mathbb{P}^n} S_{\mathcal{O}_{\mathbb{P}^n}}(\tau, z).}$$

Assume $q > 0$.

This can be shown in an elementary way, b/c know J-functions & integrals

(from last talk: $\displaystyle\int_{c-i\infty}^{c+i\infty} \Gamma(s)^N q^s\, ds$)).

(Mellin transform of Γ-function)

Can also deduce from loop space Heuristics:

LHS: $\Omega = \omega - z\mathcal{A}$ ← action functional, $\Delta \subset \mathcal{L}_{poly}\mathbb{P}^n$ poly. loops

symp. form on loop space. ← boundaries of 'hol. disks'.

$\Vert$

$\displaystyle\int_\Delta e^{\Omega/z} \underset{\text{Heuristic}}{=\!=} \text{RHS}$, & Γ class appears in RHS equality.

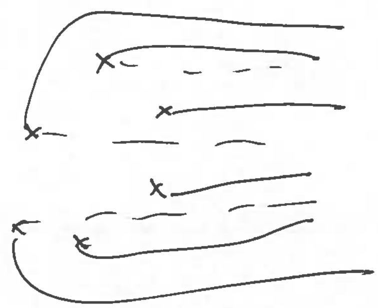
Once you know, $\Gamma_{\mathbb{R}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}$, monodromy argument $\Longrightarrow$

$$\times \text{———} \longleftrightarrow \mathcal{O}(1)$$

$$\times \longrightarrow \Gamma_{\mathbb{R}} \longleftrightarrow \mathcal{O}_{\mathbb{P}^n}$$

$$\times \longrightarrow \cdot \longleftrightarrow \mathcal{O}(-1).$$

In general, monodromy transformation goes



& use mutation to turn into straight paths.

Rmk: TEP (T: twister, E: extension, P: pairing) TERP (R: real).