Top K-theory of dg-categories.

Ref: • A-Blanc "Top K-theory ... nc-spaces..."

• summary in Kaledin's 2010 ICM

• short sketch in KKP

• Toen's lecture in Miami 2010 on Auroux's webpage

X alg variety over $\mathbb{C}$.

$H^\bullet_{dR}(X) \cong H^\bullet_{Betti}(X) = H^\bullet(X, \mathbb{C}) \supseteq H^\bullet_{\mathbb{Z}} = H^\bullet(X, \mathbb{Z})/\text{tors}.$

gives periods via $\langle L^\vee, \Omega \rangle = \int_L \Omega$, with $L^\vee \in H^\bullet_{\mathbb{Z}}$, $\Omega \in H^\bullet_{dR}$, $L \in H_\bullet(X, \mathbb{Z})$

Want: nc-origin of $H^\bullet_{\mathbb{Z}}$, integral lattice.

NOTE: For various reasons, it's been suggested that one should instead use the commensurable, $\text{im}(Ch: K^{top}(X^{an}) \to H^\bullet_{\mathbb{Q}}(X))$ ↪ iso after $\otimes\mathbb{Q}$

For LG-model, (X, f) $f: X \to \mathbb{C}$.

$H^\bullet_{dR}(X, f)_z = H^\bullet(\Omega^\bullet_X, d + z^{-1} df)$ (see Iritani)

$H^B_\bullet(X, f)_z = H_\bullet(X, \text{Re}\{W/z \ll 0\}; \mathbb{C})$.

$\cup\!\!\mid$

$\cancel{H^{\mathbb{Z}}_\bullet = H_\bullet(X, \text{Re}\{W/z \ll 0\}; \mathbb{Z})}$

$\Downarrow$

supposed to use $K^{top}(X, \text{Re}\{W/z \ll 0\})_{\mathbb{Z}}$

rationally, they are the same.

Q: How to get $K^{top}(X^{an})$ from $\text{perf}(X) = \mathcal{C}$?

Rmk: $H^\bullet_{dR}(X)((u)) \cong HP^\bullet(\mathcal{C})$.

Thm [Blanc] $\exists$ functors $K^{top}: dgcat_{\mathbb{C}} \to Sp$ (category of spectras) ($\pi_\bullet$ groups) ↪ $\exists$ a map, we will talk about groups

Satisfying

(1). $K^{top}(\mathbb{Q}^{\mathbb{C}}) = BU \times \mathbb{Z} = K^{top}(*)$ ($* = pt$)

(2) If X separated, finite type, then $K^{top}(\text{perf}(X)) = K^{top}(X(\mathbb{C})^{an})$

(3) M invariance, commutes w/ filtered colimits.

[Treumann can apply to $\text{Coh}(X)$ affine, when $\mathcal{H}om(X, Y)$ is inf dim over $\mathbb{C}$]

(4) Let $K^{alg}(\mathcal{C})$ denote alg K-theory

then $\exists$ a commutative diagram

$$\begin{tikzcd} K^{alg}(\mathcal{C}) \arrow[r, "ch_{alg}"] \arrow[d] & HC^-_\bullet(\mathcal{C}) \arrow[d] \\ K^{top}(\mathcal{C}) \arrow[r, "ch_{top}"] \arrow[d, "\cong"] & HP_\bullet(\mathcal{C}) \arrow[d, "\cong"] \\ K^{top}(X(\mathbb{C})^{an}) \arrow[r, "ch_{top}", "\cong"'] & H^{\bullet}_{dR}(X)((u)) \end{tikzcd}$$

(5) If $\mathcal{C} = Perf(X)$

2 flavours of definitions.

Defn: (1) Via the moduli of objects in $\mathcal{C}$ [Toën-Vaquié]

$\cong$ (2) Via $K_{alg}(\mathcal{C}\otimes -)$ on $\mathcal{C}$.

Let $\mathcal{M}^{\mathcal{C}} :=$ {perfect $\mathcal{C}$-mod}.

$\mathcal{M}_{\mathcal{C}} =$ {Proper $\mathcal{C}$-mod} "pseudo-perfect $\mathcal{C}$-mod"

$\mathcal{M}^{\mathcal{C}} : \in Fun(Aff_{\mathbb{C}}, sSet)$ functors. (affine scheme.)

$= \{ Spec(A) \longrightarrow N.\, w\, perf(\mathcal{C}\otimes^{\mathbb{L}}_{\mathbb{C}} A) \}$

Take the simplicial nerve. throw out morphisms which are NOT weak equivalence.

If $X = Spec(A)$

$\mathcal{M}_{\mathcal{C}}$ contains all smooth pts.

"point-like objects" in $\mathcal{C}$ or $spec(A)$ ∞-stack

$\mathcal{M}_{\mathcal{C}} : \in Fun(Aff_{\mathbb{C}}, sSet) := sPr(Aff_{\mathbb{C}})$ simplicial presheaf over affine schemes.

$:= \{ Spec(A) \longrightarrow N.\, w\, Fun_{\mathbb{C}}(\mathcal{C}^{op}, perf(A)) \}$

[Rmk] (1) If $\mathcal{C}$ smooth, $\mathcal{M}_{\mathcal{C}} \hookrightarrow \mathcal{M}^{\mathcal{C}}$. $\mathcal{C}$ proper $\mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}_{\mathcal{C}}$]

$\mathcal{C}$ smooth & proper $\mathcal{M}^{\mathcal{C}} = \mathcal{M}_{\mathcal{C}}$.

We will focus on the case $\mathcal{C}$ is smooth & proper

(2) If $\mathcal{C}$ is of "finite type" (stronger than "hom"-smooth)

then $\mathcal{M}_{\mathcal{C}}$ is "algebraic"

"finite type" means an iterated ext of d.g.a with finite many relations.

Counterexample: Take $\mathcal{C} = \mathbb{A}^1 \setminus \{\infty \# \text{ of pts}\}$, $perf(\mathcal{C})$ is smooth. NOT finite type. but smooth + proper $\Rightarrow$ finite type.

Topologize:

Top or $|\cdot|^{an}$: $\mathrm{Aff}_{\mathbb{C}}^{\text{finite type}} \longrightarrow$ top-space.

$\mathrm{Spec}(A) \longmapsto (\mathrm{Sp}(A)(\mathbb{C}))^{an}$.

$$\begin{tikzcd} \mathrm{Aff}_{\mathbb{C}} \arrow[r, "|\cdot|^{an}"] \arrow[dr, hook, "\text{Yoneda}"'] & \mathrm{Top} \\ & s\mathrm{Pr}(\mathrm{Aff}_{\mathbb{C}}) \arrow[u, dashed, "|\cdot|^{an}"'] \end{tikzcd}$$

where $|\mathcal{M}^{e}| := \underset{\mathrm{Spec}(A)\to\mathcal{M}^{e}}{\operatorname{holim}_{\to}} |\mathrm{Spec}(A)|^{an}$

Chern character maps β to u in HP.

Def^n: The semi-topological K-theory of e.

$K^{st}_{\bullet}(e) = \pi_{\bullet} |\mathcal{M} e|^{top/an}$.

↳ homotopy group.

can compute $K^{st}_{\bullet}(\ast)^{\mathrm{pt}} = \mathbb{Z} \times BU$.

In particular, $\exists \beta$, Bott element, in $\pi_2(\mathbb{Z}\times BU)$

so $K^{st}(e)$ is a $\mathbb{Z}[\beta]$-mod. &

$K^{top}(e) := K^{st}(e)[\beta^{-1}]$.

[Sendel] Before we invert, K^{st} still remember the alg structure; localization get rid of the dependence]

Expect

$$\begin{tikzcd} K^{st}(e) \arrow[r] \arrow[d, "\text{localize}"] & HC^{-}_{\bullet}(e) \arrow[d] \\ K^{top}(e) \arrow[r] & HP_{\bullet}(e) \end{tikzcd}$$

$K^{st}(e)$ a $K[\beta]$-mod; $HC^{-}_{\bullet}(e) \in \mathbb{Z}[u]$ mod

Ref: Another def^n more geometrical: Friedlander–Walker showed $K^{st}(X)[\beta^{-1}] = K^{top}(X)$. Toen ..., Blanc.

In nice cases, $K^{st}(X) = K^{top}(X)$, but $K^{st}(X) \neq K^{st}(\mathrm{perf}(X))$

want to get a group to deloop to get a spectrum. i.e. It's an infinite loop space.

NOTE: $K^{st}_{\bullet}(e)$ is a monoid, by direct sum.

Rmk: $\pi_{\bullet}|\mathcal{M}^{e}|^{top}$ is a group.

sketch: $E \oplus E[1]$ deforms continuously to $E \xleftarrow{id} E[1] \simeq 0$. $\exists$ an alg path $\mathbb{A}^1$ in moduli of e-mod

So $E[1] = [E]^{-1}$ in $\pi_{\bullet}|\mathcal{M}e|^{top}$.

$\exists$ Chern character: via the 2^{nd} construction of $K^{st}(X)$.

"Nerve" (simplicial bar constructions)

§2 Relation to $K_{alg}(\mathcal{C})$ — weak equiv.

Recall: $K_{alg}^{\bullet}(\mathcal{C}) = \pi_{\bullet}\Omega|N_{\bullet}wS_{\bullet}\mathcal{C}|$

loop it! — $S_k\mathcal{C}$ = "k-step filtration in $Perf(\mathcal{C})$".

$\exists$ natural map $K_{alg}^{\bullet}(\mathcal{C}) \to HC_{\bullet}^{-}(\mathcal{C})$

Ref. [Connes, Karoubi, Goodwillie, Hood-Jones, Jones ...].

Blanc: $\exists$ a presheaf of alg K-theories, as a functor, it's

$\underline{K}: Spec(A) \longrightarrow K_{alg}^{\bullet}(\mathcal{C}\otimes_{\mathbb{C}}^{L} A)$

then $K_{\bullet}^{st}(\mathcal{C}) := |\underline{K}(\mathbb{C})|^{top}$

why? have a map $\mathcal{M}^{\mathcal{C}} \hookrightarrow \underline{K}(\mathcal{C})$

e.g. $a_1\oplus a_2\oplus a_3 \mapsto (a_1 \hookrightarrow a_1\oplus a_2 \hookrightarrow a_1\oplus a_2\oplus a_3)$ → map to trivial filtration.

induces $|\mathcal{M}^{\mathcal{C}}|^{top} \simeq |\underline{K}(\mathcal{C})|$

↳ b/c any non-trivial filtration admits a continuous (or $\mathbb{A}^1$-) homotopy to the trivial one (as previously).

$\exists$ an inherited chern character.

$$\begin{array}{ccccc}
K_{alg}^{\bullet}(\mathcal{C}) = \underline{K}(\mathcal{C}) & \to & |\underline{K}|^{top} = K^{st}(\mathcal{C}) & \longrightarrow & K^{top}(\mathcal{C}) \\
\downarrow ch_{alg} & & \downarrow (ch_{alg})^{top} & & \downarrow (ch_{alg})^{top}[\beta^{-1}] \\
HC_{\bullet}^{-}(\mathcal{C}) & \longrightarrow & \underbrace{|Spec(A) \to HC^{-}(\mathcal{C}\otimes^{L} A)|}_{\text{want to call it } H_{B}^{\bullet}(\mathcal{C})} & \longrightarrow & |Spec(A) \to HP_{\bullet}(\mathcal{C}\otimes^{L} A)|^{top}
\end{array}$$

[Kassel] SII Künneth $HP^{\bullet}(\mathcal{C})\otimes^{L}_{\mathbb{C}[u,u^{-1}]} HP^{\bullet}(A)$

Also, $|Spec(A) \to HP^{\bullet}(\mathcal{C}\otimes^{L} A)|^{top}$

$\simeq HP^{\bullet}(\mathcal{C}) \otimes \underbrace{|Spec(A) \to HP^{\bullet}(A)|^{top}}_{HP^{\bullet}(pt) = k[u^{\pm 1}]} = HP^{\bullet}(\mathcal{C})$

[Ref: Iritani on TERP structure.]

$\exists$ a nc-period map $HP^{\bullet}(\mathcal{C}) \otimes K_{\bullet}^{top}(\mathcal{C}) \longrightarrow \mathbb{K}$

↑ K-homology

Aside: Derived equiv does NOT preserve $\mathbb{Z}$-Lattice, but preserve the K-theory lattice.