

1) why k-theory?

2) moduli of objects in dg category

3) top. k-theory of dg categories, + top. k-theory.

4) relation to algebraic k-theory & the Chern character.

References: • Blanc, "Top. --"

• Kaledin ICH address

• KKP,

• Kontsevich, Solomon-Lefschetz

• Waldhausen, etc

X alg variety / \mathbb{C} or $\mathbb{C}[t]$.

$\rightarrow H_{dR}^i(X) = H^i(\Omega_X^i, d) \cong H_B^i(X) = H^i(X, \mathbb{C}) \cong H_{\mathbb{Z}}^i(X) = H^i(X, \mathbb{Z}) / \text{tors.}$
↑ integral structure

(or LG pair (X, W) more related to situation we've been studying. gives periods via $\int_{\gamma} \Omega$ (hol. vol. form) = $\langle L, \Omega \rangle$
↑ integral cycle. $L \in H^i(X, \mathbb{Z})$

$H_{dR}^i(X, W) = H_{\mathbb{R}}^i(\Omega_X^i, d + \pm^{-1} dW)$

periods \leadsto "oscillating integrals".

$H_B^i(X, W) = H^i(X, \{ \text{Re}(W(t)/z) < 0 \}; \mathbb{C})$

$L \in H^i(X, \mathbb{Z})$

$H_{\mathbb{R}}^i(X, W) = H^i(X, \{ \text{Re}(W(t)/z) < 0 \}; \mathbb{R})$

$\Rightarrow \mathbb{R}(L)$

Expectations(?): $H^i(X, \mathbb{Z})$ is poorly behaved as a non-commutative invt. (invt. of category $\text{perf}(X)$) δ
 should be replaced by $K^{top}(X)$. (E.g. case: (1) homology inst. of algebraic \sim (2) LG case: what $K_{top}(X, \{ \text{Re}(W(t)/z) < 0 \})$?)

• Has a different lattice $ch: K^{top}(X) \rightarrow H_B^i(X, \mathbb{C})$ which is commensurate w/ $H^i(X, \mathbb{Z})$ & better behaved.

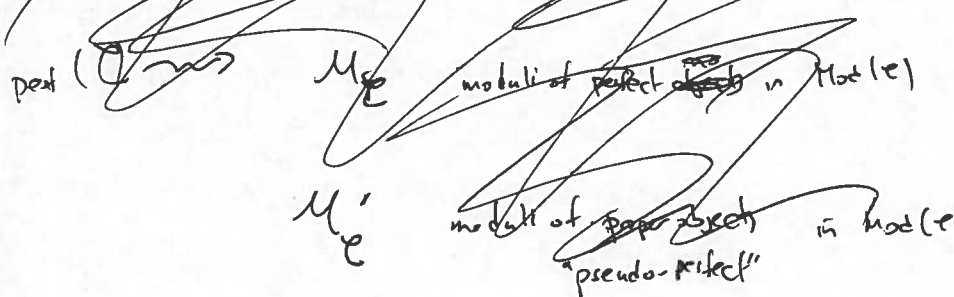
How to get $K^{top}(X)$ from $\text{perf}(X)$? [Blanc, following proposal of Toen]

Caveats: we will be sketching about spectra vs. spaces vs. chain complexes (vs. \mathbb{Q} .)

Already know $H_{dR}^i(X) \cong H_{HKR}^i(X) \cong HP^i(\text{perf}(X))$

Ea dg/A ∞ category / \mathbb{C} \leftarrow very important.

1) Moduli of objects in \mathcal{C} [Toën-Vaquié].



Thm [Blanc]: \exists a functor $K^{top} = dg \text{Cat}_{\mathbb{C}} \rightarrow \text{Spectra}_{\text{ccs}}$, called topological k-theory of ncg -paces.

interesting:

(a) $K^{top}(\mathbb{C}) \cong BU = K_{top}(\mathbb{C})$.

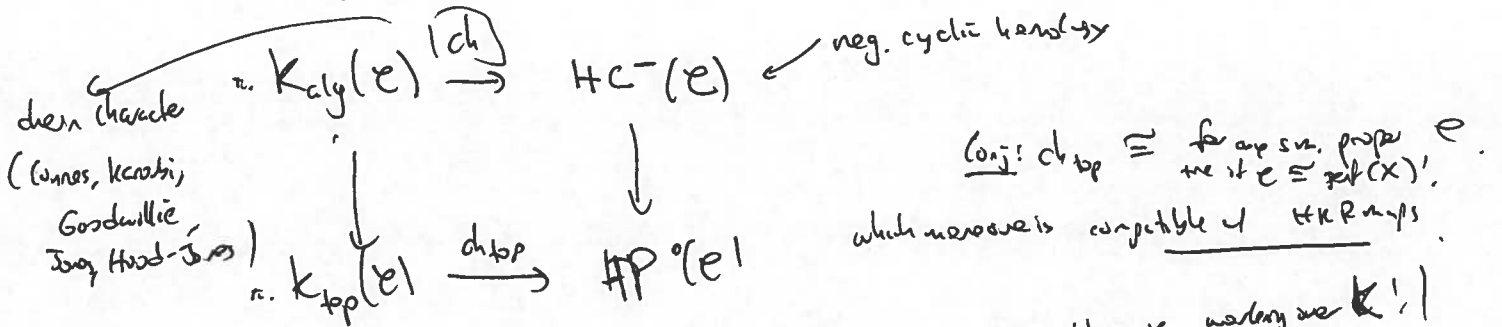
(b) If X sep. \mathbb{C} -scheme finite type,

$$K_{\text{top}}(\text{perf}(X)) \cong K_{\text{top}}(X(\mathbb{C}))$$

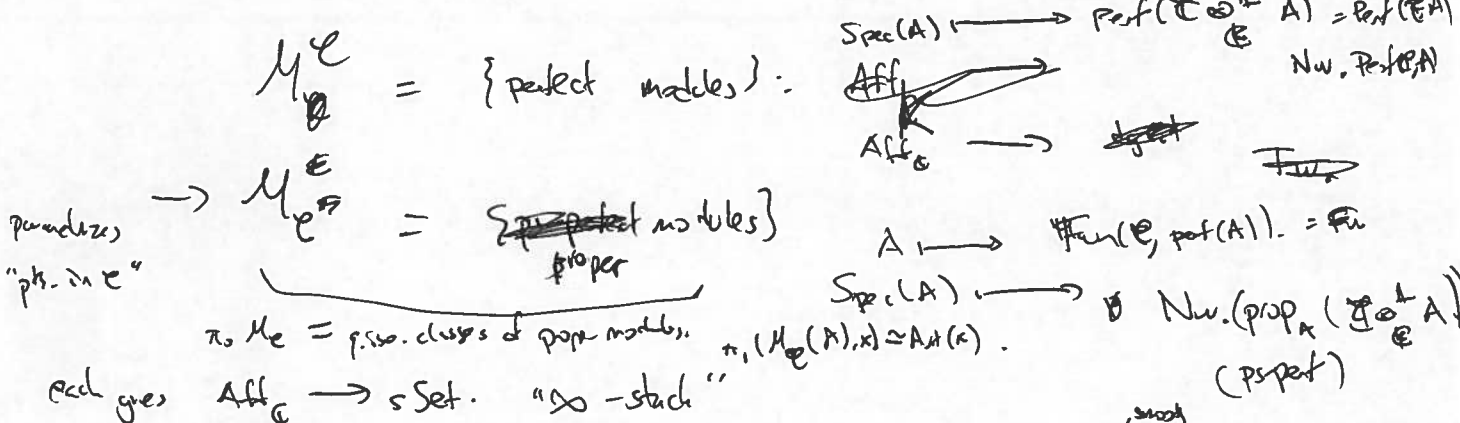
or rather
 $Q_{\text{coh}}(X)$

(e) Mordell-Weil, constants \mathbb{C} (filled) algebras, etc.

(d) let K_{alg} denote algebraic k -theory. then, \exists a functorial diagram



Definition 1) \mathcal{E} dg cat / \mathbb{K} its moduli of objects $\cong \text{Mod}(\mathcal{E})$ [Toën-Vaquié], Actually too interesting moduli spaces



Facts: 1) if \mathcal{E} is of "finite type" (slightly stronger than smooth, holds for $\text{perf}(X)$ if X has fin. type).

$\Rightarrow M_{\mathcal{E}}$ algebraic "locally algebraic but finite moduli".

2) \mathcal{E} ~~prop~~ smooth: $M_{\mathcal{E}} \hookrightarrow M_{\mathcal{E}}$

- prop $M_{\mathcal{E}} \hookrightarrow M_{\mathcal{E}}$

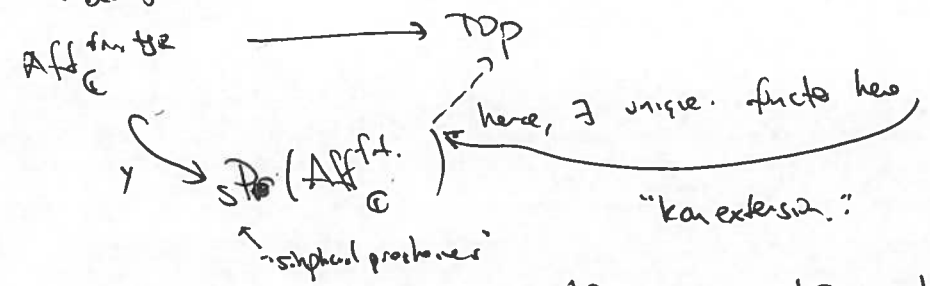
3) \mathcal{T} smooth + prop = $M_{\mathcal{E}} \cong M_{\mathcal{E}}$

(ex: $\mathcal{E} = A\text{-mod}$: $M_{\mathcal{E}}^{\text{prop}}: \text{Spec } B \rightarrow A \otimes B\text{-modules}$ projective of finite rank over B .)

represent as: $\cong \coprod_{n \in \mathbb{N}} \text{Hom}_{k\text{-alg}}(A, M_n(k)) / GL_n(k)$ (non-abelian)

2) topologize: Claim: $K\text{-semi-top } P(\mathcal{C}) := \text{real } |\mathcal{C}|^{\text{top}}$ "underlying topological space"
 (semi-top K -modulay (closely related to Friedlander-Waller's "semi-topological K-theory")
 Have $\text{Top} = |\cdot|^{\text{an}}$: $\text{Aff}_{\mathbb{C}}^{\text{fin. type.}} \rightarrow \text{Top Spaces} = K\text{-Mod}_{\mathbb{C}}$
 $\text{Spec } R \mapsto \text{Spec } R(\mathbb{C})^{\text{top}}$ as a top. space, w/ top. inherited from \mathbb{C}^n .

now, the Yoneda embedding induces



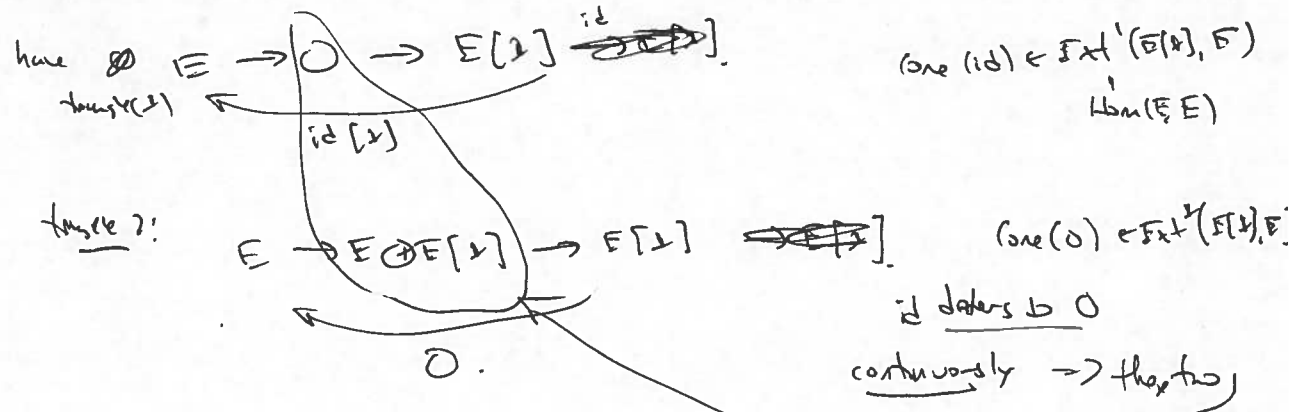
if $F: \text{Aff}_{\mathbb{C}}^{\text{f.t.}} \rightarrow s\text{Set}$, then, $|F|^{\text{top}} = \text{hom}_{\text{Top}}(|\text{Spec } A|^{\text{top}}, \text{Spec } A \rightarrow F)$ "Bott periodicity"

~~Imp~~ $F \mapsto |F|^{\text{top}}$ sends local isomorphisms to weak equivalences.

a) $|\mathcal{M}_E|$ is a monoid by direct sum

(b) $\pi_0 |\mathcal{M}_E|$ is actually a group! ~~not true for any spec~~ (hence can deloop, get a spectrum)
 sketch:

Not for any \mathbb{C} -module E , $[E \oplus E[2]] = 0$ in $\pi_0(|\mathcal{M}_E|)$.



represent the same point.

Lem: $k^{\text{semi-top}}(\text{pt.}) = BU \simeq \mathbb{Z} \times BU$
 $(\coprod_{n \geq 0} BGL_n)^+$

so every $k^{\text{st}}(e)$ is a graded $\mathbb{Z}[\beta]$ -module, coming from $\pi_2 BU = \langle \beta \rangle$.

Def: $k^{\text{top}}(e) = k^{\text{semi-top}}(e)[\beta^{-1}]$.

sketch: ~~Def: $\text{Vect}(\mathbb{C}) \simeq$ moduli of vector spaces $\simeq \coprod_{n \geq 0} BGL_n$ (for matrix of $n \times n$ complex sq.)~~
 $\text{Vect.} \simeq \coprod_{n \geq 0} BGL_n$

now, $BU \simeq \text{group } |\text{Vect.}|^+$
 $\simeq (\coprod_{n \geq 0} BGL_n(\mathbb{C}))^+$
 $= (\coprod_{n \geq 0} BU \wedge \mathbb{C}P^n)^+ = BU \times \mathbb{Z}$.

$\pi_0 = BU$.

then check:

3) relation to algebraic K-theory:

have $k_{\text{alg}}^i(e) = (\pi_0) \Omega |NW. S_e|$
 \uparrow \uparrow
 vec. equib. $S_k e$ is the set of "k-step filtrations in e."

δ = natural map

$k_{\text{alg}}^i(e) \xrightarrow{\text{ch.}} HC^-(e)$ [Cous, Karasbi, Goodwillie, Thomason, Dwyer]

for $k^0(e)$

$\text{ch}(f_x) = f_x^* \left(\begin{matrix} 1 & 0 \\ 0 & x \end{matrix} \right)$ in the functor

$k \xrightarrow{f_x} e$
 $f_x^* : HC^-(k) \rightarrow HC^-(e)$
 \simeq
 $k[u]$
 $1u^0$

'Blanc': (can actually define

Tabuada, Cisinski - Tabuada.

Schlichting.

$K_{\text{semi-top}}(e)$ via algebraic K-theory as follows:

there is a presheaf of algebraic K-theories: ^(non-connective)

$$\underline{K}(e): \text{Aff}_{\mathbb{C}}^{\text{op}} \rightarrow S_p$$

$$(\text{Spec } A) \mapsto K(\mathbb{C} \otimes_{\mathbb{A}} A).$$

then $K_{\text{semi-top}}(e)$ also $\simeq | \underline{K}(e) |^{\text{top}}$.

n.b. exists a canonical map

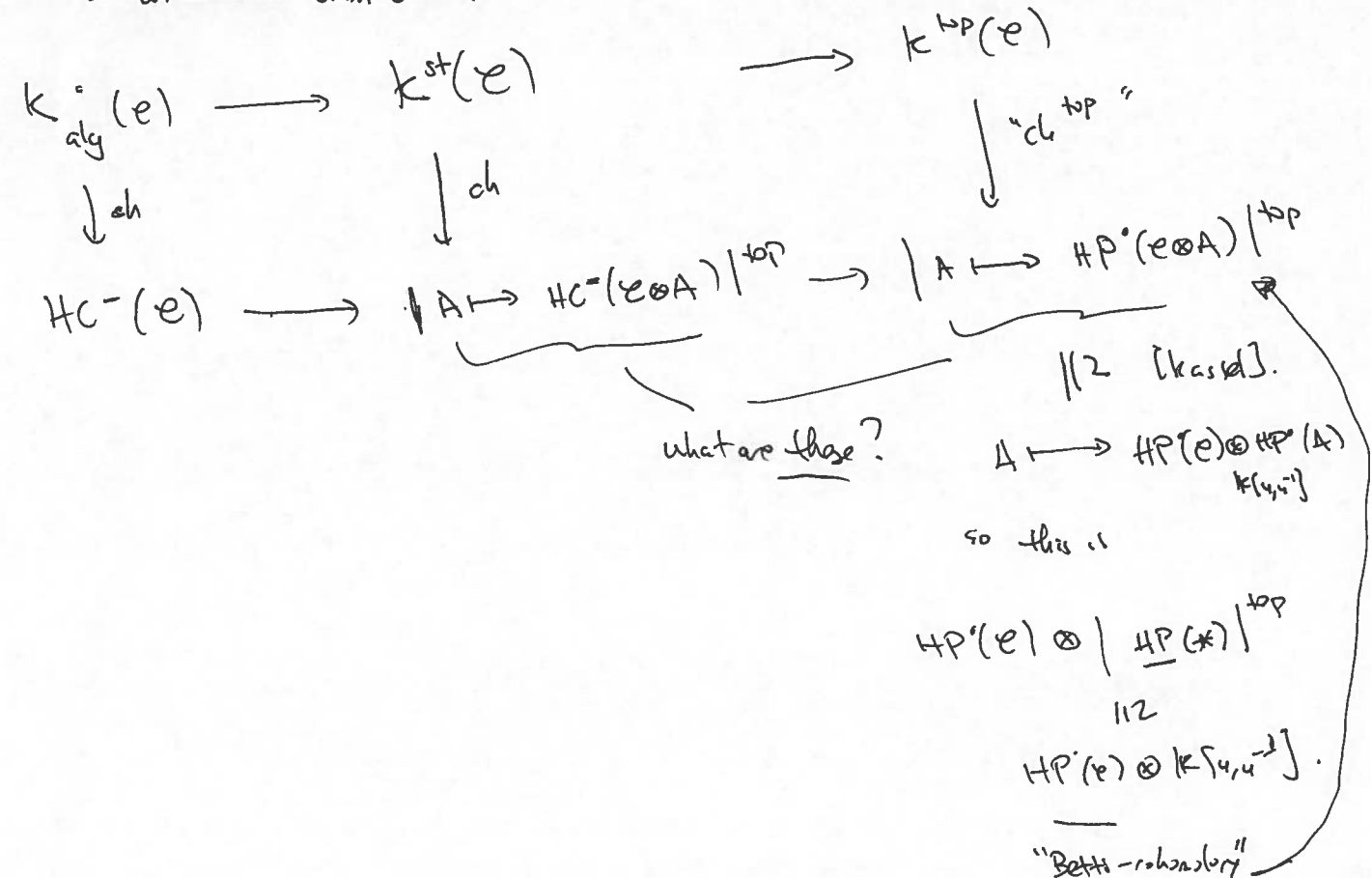
$$K_{\text{alg}}^{\bullet}(T) \rightarrow K^{\text{st}}(T).$$

why? Have $\mathcal{M}_e \hookrightarrow \underline{K}(e)$ inclusion of presheaves

$$a_1 \otimes a_2 \otimes a_3 \hookrightarrow a_1 \hookrightarrow a_1 \otimes a_2 \otimes a_3 \hookrightarrow a_1 \otimes a_2 \otimes a_3$$

which induces an A^1 homotopy-equivalence "can smoothly deform any filtration until it's totally split", hence an equivalence on $| - |^{\text{top}}$.

hence, there is an inherited Chern character:



Relation w/ commutative case:

$$\pi_0 \text{ ssp}(F) = F(\mathbb{C}) / \sim$$

where $[x] \sim [y]$ if exists a

conn. alg. curve, a map

$$f: h_C \rightarrow F \text{ in } \text{Ho}(\text{Sp}(\mathbb{A}_1))$$

to cpl. pts.

$$x', y' \in C(\mathbb{C})$$

$$\text{s.t. } f(x') = x$$

$$f(y') = y.$$

monogonally

ssp sends A^1 -htpy

equivalences to

htpy-equivalences.

Fredbecker-Waller is $\underline{K}_0^{\text{semi-top}}(X) = \text{Alg. maps}(X, \mathbb{Z}^x, \text{BU})$.

alg. vec. bundles / $\frac{\text{alg.}}{\text{equivalence}}$

e.g. $V_0 \cong_{\text{alg}} V_2$ ↙ curve

if $\exists V$ on $C \times X$

$$\text{w/ } V|_{\text{pt} \times X} = V_0, \quad V|_{b \times X} = V_2.$$

thm: $\underline{K}(X) \cong \underline{K}(\text{pt} \times X)$ (though this is technical for).

so, need to compare

$$(\underline{K}(X) / \text{top}) \quad \text{vs} \quad K_{\text{top}}(X^{\text{an}})$$

$$\text{RHom}_{\text{Ho}(\text{Sp})}(\sum_{S^1} |X|_+, \text{BU}) ?$$

Butt-inverted,

Remarks: 1) Thomason already showed the "étale sheafification of $K_{\text{alg}}(X)$ was $K^{\text{top}}(X)$

at least in finite characteristic"

2) more general results of Fredbecker-Waller, Cohen-Lima-Filho

3) Blanc's proof is machinery heavy