0) why K-theory?

1) moduli of objects in a dg category

2) semi-top. K-theory of dg categories, + top. K-theory.

3) relation to algebraic K-theory & the Chern character.

References: • Blanc, "Top. — —"
• Kaledin ICM address
• KKP,
• Kontsevich, Soibelman–Lefschetz notes
• Waldhausen, etc.

X alg variety $/\mathbb{C}$ or $\mathbb{C}[[t]]$.

$\rightsquigarrow H_{dR}^\cdot(X) = \mathbb{H}^\cdot(\Omega_X^\cdot, d) \cong H_B^\cdot(X) = H^\cdot(X, \mathbb{C}) \supseteq H_{\mathbb{Z}}^\cdot(X) = H^\cdot(X, \mathbb{Z})/\text{tors}.$

$\uparrow$ integral structure

(or LG pair (X,W), more related to situation we've been studying. gives periods via $\int_L \Omega$ ← hol. vol. form $= \langle L^\vee, \Omega \rangle$
← integral cycle. $L^\vee \in H^\cdot(X, \mathbb{Z})$. $PD(L)$.

$\mathfrak{z} = H_{dR}^\cdot(X,W)_z = \mathbb{H}^\cdot(\Omega_X^\cdot, d + z^{-1} dW)$ periods $\rightsquigarrow$ "oscillating integrals".

$H_B^\cdot(X,W)_z = H^\cdot(X, \{Re(W(y)/z) \ll 0\}; \mathbb{C})$.

$H_{\mathbb{Z}}^\cdot(X,W)_z = H^\cdot(X, \{Re(W(y)/z) \ll 0\}; \mathbb{Z})$)

Expectations(?): $H^\cdot(X, \mathbb{Z})$ is poorly behaved as a non-commutative invt. (invt. of category perf(X)) & should be replaced by $K^{top}(X)$. (Remarks: 1) homology instead of cohomology 2) LG case: want $K_{top}(X, \{Re(W(y)/z) \ll 0\})$)

, have a different lattice $ch: K^{top}(X) \longrightarrow H_B^\cdot(X, \mathbb{C})$ which is commensurable w/ $H^\cdot(X, \mathbb{Z})$, & better behaved.

How to get $K^{top}(X)$ from perf(X)? [Blanc, following proposal of Toën]. Caveats: we will be sketchy about spectra vs. spaces vs. chain complexes ^ vs. $\otimes$.)

$\mathcal{C}$

Already know $H_{dR}^\cdot(X)((u)) \underset{HKR}{\cong} HP^\cdot(\text{perf}(X))$.

$\mathcal{C}$ a dg/A_∞ category $/\mathbb{C}$ ← very important.

1) ~~Look at the moduli of objects in $\mathcal{C}$ [Toën–Vaquié].~~

~~perf$(\mathcal{C}) \rightsquigarrow \mathcal{M}_{\mathcal{C}}$ moduli of perfect objects in Mod$(\mathcal{C})$~~

~~$\mathcal{M}'_{\mathcal{C}}$ moduli of proper objects in Mod$(\mathcal{C})$. "pseudo-perfect"~~

Thm [Blanc]: $\exists$ a functor $K^{top}: dgCat_{\mathbb{C}} \longrightarrow$ Spectra (spaces) ($\xrightarrow{\pi_i}$ groups), called topological K-theory of nc-spaces.

satisfying:

(a) $K^{top}(\mathbb{C}) \simeq BU = K_{top}(*)$.

(b) If X sep. $\mathbb{C}$-scheme finite type,

$$K^{top}(\mathrm{perf}(X)) \cong K_{top}(X(\mathbb{C}))$$

or rather $\mathrm{Qcoh}(X)$

(c) Morita invt, commutes w/ (filtered) colimits, etc.

(d) Let K_{alg} denote algebraic K-theory. Then, $\exists$ a functorial diagram

$$\begin{tikzcd} \pi_* K_{alg}(\mathcal{C}) \arrow[r, "ch"] \arrow[d] & HC^-(\mathcal{C}) \arrow[d] \\ \pi_* K_{top}(\mathcal{C}) \arrow[r, "ch_{top}"] & HP^\circ(\mathcal{C}) \end{tikzcd}$$

neg. cyclic homology

chern character (Connes, Karoubi, Goodwillie, Jones, Hood-Jones)

Conj: $ch_{top} \cong$ for any sm. proper $\mathcal{C}$. true if $\mathcal{C} \cong \mathrm{perf}(X)$, which moreover is compatible w/ HKR maps.

(doesn't require working over $\mathbb{C}$!)

Definition(s) $\mathcal{C}$ dg cat $\rightsquigarrow$ its moduli of objects [Toën-Vaquie]. Actually two interesting moduli spaces

in $\mathrm{Mod}(\mathcal{C})$

$$\mathcal{M}^{\mathcal{C}} = \{\text{perfect modules}\}: \quad \mathrm{Aff}_{\mathbb{C}} \longrightarrow$$

$$\mathrm{Spec}(A) \longmapsto \mathrm{Perf}(\mathcal{C} \otimes^L_{\mathbb{C}} A) = \mathrm{Perf}(\mathcal{C}, A)$$

$\mathrm{Nw}_\cdot\,\mathrm{Perf}(\mathcal{C}, A)$

parametrizes "pts. in $\mathcal{C}$" $\longrightarrow$

$$\mathcal{M}_{\mathcal{C}} = \{\text{proper modules}\}$$

$A \longmapsto \mathrm{Fun}(\mathcal{C}, \mathrm{perf}(A))$.

$$\mathrm{Spec}(A) \longmapsto \mathrm{Nw}_\cdot(\mathrm{prop}_A(\mathcal{C} \otimes^L_{\mathbb{C}} A))$$

(ps-perf)

$\pi_0 \mathcal{M}_{\mathcal{C}}$ = g.iso. classes of proper modules, $\pi_1(\mathcal{M}_{\mathcal{C}}(A), x) \simeq \mathrm{Aut}(x)$.

each gives $\mathrm{Aff}_{\mathbb{C}} \longrightarrow s\mathrm{Set}$. "$\infty$-stack"

Facts:

1) if $\mathcal{C}$ is of "finite type" (slightly stronger than smooth, holds for $\mathrm{perf}(X)$ iff X has fin. type) smooth

$\Rightarrow \mathcal{M}^{\mathcal{C}}$ algebraic "locally geometric & of finite presentation".

2) $\mathcal{C}$ smooth: $\mathcal{M}_{\mathcal{C}} \hookrightarrow \mathcal{M}^{\mathcal{C}}$

- proper $\mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}_{\mathcal{C}}$

3) T smooth & proper: $\mathcal{M}^{\mathcal{C}} \cong \mathcal{M}_{\mathcal{C}}$.

(ex: $\mathcal{C} = A\text{-mod}$: $\mathcal{M}_{\mathcal{C}}^{\mathcal{C}}$: $\mathrm{Spec}\,B \longrightarrow A \otimes B$-modules projective of finite rank over B.

represent as: $\coprod_{n \in \mathbb{N}} \mathrm{Hom}_{k\text{-alg}}(A, M_n(k)) / GL_n(k)$. (non-free action.)

2) topologize: **Claim:** $K^{\text{semi-top}}(\mathcal{C}) := \pi_* |\mathcal{M}_{\mathcal{C}}|^{top}$ ← "underlying topological space" of semi-top K-theory (closely related to Friedlander-Walker's "semi-topological K-theory")

Have $\text{Top} = |\cdot|^{an}: \text{Aff}_{\mathbb{C}}^{\text{fin.type}} \longrightarrow \text{Top Spaces}$ $= K_* \pi_* |\mathcal{M}_{\mathcal{C}}|$.

$\text{Spec} R \longmapsto \text{Spec} R(\mathbb{C})^{top}$, as a top. space, w/ top. inherited from $\mathbb{C}^n$.

now, the Yoneda embedding induces

$$\begin{tikzcd} \text{Aff}_{\mathbb{C}}^{\text{fin. type}} \arrow[r] \arrow[d, hook, "y"'] & \text{Top} \\ s\text{Pre}(\text{Aff}_{\mathbb{C}}^{f.t.}) \arrow[ur, dashed] & \end{tikzcd}$$

here, ∃ unique functor here

"Kan extension."

↖ "simplicial presheaves"

if $F: \text{Aff}_{\mathbb{C}}^{f.t.} \longrightarrow s\text{Set}$, then, $|F|^{top} = \underset{\text{Spec} A \to F}{\text{hocolim}} |\text{Spec} A|^{top}$ "Betti realization"

Imp. ~~Exercise~~ $F \longmapsto |F|$ sends local isomorphisms to weak equivalences.

a) $|\mathcal{M}_{\mathcal{C}}|$ is a monoid by direct sum

(b) $\pi_0 |\mathcal{M}_{\mathcal{C}}|$ is actually a group! ~~[illegible]~~ (hence can deloop, get a spectrum

sketch:

Note for any $\mathcal{C}$-module E, $[E \oplus E[1]] = 0$ in $\pi_0(|\mathcal{M}_{\mathcal{C}}|)$.

have $E \to 0 \to E[1] \xrightarrow{id} ~~[illegible]~~]$. (one (id) $\in \text{Ext}^1(E[1], E)$ $= \text{Hom}(E, E)$

triangle(1) $\quad$ id $[1]$

triangle 2: $E \to E \oplus E[1] \to E[1] \quad ~~[illegible]~~]$. (one (0) $\in \text{Ext}^1(E[1], E)$

0.

id deforms to 0 continuously → the two

represent the same point.

Lem: $K^{\text{semi-top}}_{\bullet}(\text{pt.}) \overset{?^{\mathbb{C}}}{=} \cancel{\ } bu \simeq \mathbb{Z} \times BU$

$(\coprod_n BGL_n)^+$.

So every $K^{st}_{\bullet}(\mathcal{C})$ is a graded $\mathbb{Z}[\beta]$-module, coming from $\pi_2 BU = \mathbb{Z}\langle\beta\rangle$.

Def: $K^{top}_{\bullet}(\mathcal{C}) = K^{\text{semi-top}}_{\bullet}(\mathcal{C})[\beta^{-1}]$.

sketch: ~~Perf(k) ≃ moduli of vector spaces~~ $\cancel{\coprod_{n\geq 0} BGL_n}$ ~~(...)~~

$\text{Vect}_{\bullet} \simeq \coprod BGl_n$.

now, $|\mathcal{M}_{\text{Vect}_{\bullet}}| \overset{\text{group}}{=} |\text{Vect}_{\bullet}|^+ = (\coprod BGl_n(\mathbb{C}))^+$

$= (\coprod BU_n(\mathbb{C}))^+ = BU \times \mathbb{Z}$.

$\pi_0 = BU$.

chern character:

3) relation to algebraic k-theory:

have $K^{\bullet}_{alg}(\mathcal{C}) = (\pi_0)\, \Omega |Nw_{\bullet} S_{\bullet} \mathcal{C}|$

weak equivalences

$S_k \mathcal{C}$ is the set of "k-step filtrations in $\mathcal{C}$".

& a natural map

$$K^{\bullet}_{alg}(\mathcal{C}) \xrightarrow{ch.} HC^-(\mathcal{C})$$ [Connes, Karoubi, Goodwillie, Hood Jones, ?]

for $K^0(\mathcal{C})$

$ch([X]) = f^X_*(1^0_u)$ in the functor

$\mathbb{K} \xrightarrow{f_X} \mathcal{C}$

$f^X_*: HC^-(\mathbb{K}) \longrightarrow HC^-(\mathcal{C})$

$HC^-(\mathbb{K}) = \mathbb{K}[u]$

$1u^0$

"Blanc": can actually define

Tabuada, Cisinski – Tabuada.

Schlichting.

$K_\bullet^{semi-top}(\mathcal{C})$ via algebraic K-theory as follows:

there is a pre-sheaf of (non-connective) algebraic K-theories:

$$\underline{K}(\mathcal{C}): \mathrm{Aff}_{\mathbb{C}}^{op} \to Sp$$
$$(\mathrm{Spec}\, A) \mapsto K(\mathcal{C} \otimes_{\mathbb{C}}^{\amalg} A).$$

then $K_\bullet^{semi-top}(\mathcal{C})$ also $\simeq |\underline{K}(\mathcal{C})|^{top}$.

n.b. exists a canonical map $K_{alg}^{\bullet}(T) \to K^{st}(T)$.

why? have $\mathcal{M}_{\mathcal{C}} \hookrightarrow \underline{K}(\mathcal{C})$ inclusion of presheaves

$$a_1 \oplus a_2 \oplus a_3 \longmapsto a_1 \hookrightarrow a_1 \oplus a_2 \hookrightarrow a_1 \oplus a_2 \oplus a_3$$

which induces an $\mathbb{A}^1$ homotopy-equivalence "can smoothly deform any filtration until it's totally split", hence an equivalence on $|-|^{top}$.

hence, there is an inherited chern character:

$$\begin{tikzcd}
K_{alg}^{\bullet}(\mathcal{C}) \arrow[r] \arrow[d, "ch"] & K^{st}(\mathcal{C}) \arrow[rr] \arrow[d, "ch"] & & K^{top}(\mathcal{C}) \arrow[d, "\text{“}ch^{top}\text{”}"] \\
HC^-(\mathcal{C}) \arrow[r] & \underbrace{|A \mapsto HC^-(\mathcal{C}\otimes A)|^{top}}_{\text{what are these?}} \arrow[rr] & & \underbrace{|A \mapsto HP^\bullet(\mathcal{C}\otimes A)|^{top}}
\end{tikzcd}$$

$\cong$ [Kassel].

$A \mapsto HP^\bullet(\mathcal{C}) \otimes_{k[u,u^{-1}]} HP^\bullet(A)$

so this is

$$HP^\bullet(\mathcal{C}) \otimes |\underline{HP}(*)|^{top}$$
$$\cong$$
$$HP^\bullet(\mathcal{C}) \otimes k[u,u^{-1}].$$

"Betti-cohomology"

Relation w/ commutative case:

Friedlander-Walker's $K_0^{\text{semitop}}(X) = \text{Alg. maps}(X, \overset{\mathbb{Z}\times}{BU})\ldots$

alg. vec. bundles / $\frac{\text{alg.}}{\text{equivalence}}$

e.g. $V_0 \simeq_{alg} V_1$

if $\exists$ V on $C \times X$ (curve)

w/ $V|_{a \times X} = V_0$, $V|_{b \times X} = V_1$.

Here: $\underline{K}(X) \cong \underline{K}(\text{perf}(X))$ (though this is technical fact).

so, need to compare

$(\underline{K}(X))^{\text{top}}[\beta^{-1}]$ w/ $K_{top}(X^{an})$.

$$= \underline{RHom}_{Ho(Sp\Sigma?)}(\Sigma^\infty_{S^1}|X|_+, BU)\ ?$$

Bott-inverted,

Remarks: 1) Thomason already showed the "étale sheafificate of $K^{alg}(X)$" was $K^{top}(X)$ at least in finite characteristic"

2) more general results of Friedlander-Walker, Cohen-Lima-Filho

3) Blanc's proof: is machinery heavy

Cohen-Lima-Filho

$\pi_0\, ssp(F) = F(\mathbb{C})/\sim$

where $[x] \sim [y]$ if exists a conn. alg. curve, a map $f: h_C \to F$ in $Ho(Sp(\text{Aff}))$ & two cplx. pts. $x', y' \in C(\mathbb{C})$ s.t. $f(x') = x$, $f(y') = y$.

more generally ssp sends $\mathbb{A}^1$-htpy equivalence to htpy-equivalences.