(a)(i)

These are notes on the talk for Gamma conjectures reading group.

1. Quantum connection

Throughout the notes let F be a Fano manifold. Let $H^{even} := H^{even}(F, \mathbb{C})$ be the even cohomology of F. It comes with a nondegenerate Poincare pairing $\langle \cdot, \cdot \rangle$. For a homogeneous element $\beta \in H^{2p} \subset H^{even}$ we denote $\deg(\beta) = p$. Consider the family of products $\star_\tau$ on H^{even}, indexed by $\tau \in H^{even}$, defined by the pairings with arbitrary elements of H^{even}

$$\langle \alpha_1 \star_\tau \alpha_2, \alpha_3 \rangle_F := \sum_{d \in \mathrm{Eff}(F)} \sum_{n \geq 0} \frac{1}{n!} \langle \alpha_1, \alpha_2, \alpha_3, \tau, \ldots, \tau \rangle_{0,3+n,d}$$

where the sum is taken over all effective curve classes d and

$$\langle \alpha_1, \alpha_2, \alpha_3, \tau, \ldots, \tau \rangle_{0,3+n,d} = \int_{[M_{0,3+n,d}]^{virt}} ev_1^*(\alpha_1) \cdot ev_2^*(\alpha_2) \cdot ev_3^*(\alpha_3) \cdot ev_4^*(\tau) \cdot \ldots \cdot ev_{n+3}^*(\tau)$$

is genus 0 Gromov-Witten invariant for the pullbacks of the evaluation maps to the moduli space of genus zero stable curves C with $n+3$ markings and $f_*[C] = d$.

Remark 1.1. Alternatively, we can use pushforwards of evaluation maps to define

$$\alpha_1 \star_\tau \alpha_2 = \sum_{d \in \mathrm{Eff}(F)} \sum_{n \geq 0} \frac{1}{n!} (ev_3)_* ([M_{0,3+n,d}]_{virt} \cap ev_1^*(\alpha_1) \cdot ev_2^*(\alpha_2) \cdot ev_4^*(\tau) \cdot \ldots \cdot ev_{n+3}^*(\tau)).$$

Because the variety is Fano, the convergence issues are not too bad, but still not guaranteed. I am going to ignore them, to avoid introducing Novikov ring. At any rate for $\tau = 0$ the product is already non-trivial, but the convergence is not in question, because only a finite number of d would contribute. Specifically, if $c_1(F) \cdot d$ is large enough, the (virtual) dimension of $M_{0,3,d}$ is too large so the GW invariant is zero.

A standard result in GW theory is that these products are commutative and associative. Commutativity is obvious from the definitions, and associativity eventually boils down to a relation in the Chow ring of $M_{0,4}$.

One way to encode this data is by introducing an auxiliary variable z (a coordinate on $\mathbb{CP}^1$) and the (meromorphic) *big quantum* connection on the trivial bundle $H^{even} \times (H^{even} \times \mathbb{CP}^1) \to (H^{even} \times \mathbb{CP}^1)$

$$\nabla_{z\partial_z} := z\frac{\partial}{\partial z} - \frac{1}{z}(E \star_\tau) + \mu, \qquad \nabla_{\partial_\alpha} := \frac{\partial}{\partial \alpha} + \frac{1}{z}(\alpha \star_\tau)$$

where μ is the grading operator, which acts by $(p - \frac{1}{2}\dim F)$ on $H^{2p}(F, \mathbb{C}) \subset H^{even}$ and

$$E = c_1(F) + (1 - \frac{1}{2}\dim F - \mu)(\tau).$$

We will be especially interested in the part of the connection for $\tau = 0$, which gives

$$\nabla_{z\partial_z} = z\frac{d}{dz} - \frac{1}{z}(c_1(F)\star_0) + \mu.$$

Exercise.[1] Verify various properties of this connection. Specifically, check that it is flat and that it is compatible with Poincare pairing in the sense of

$$\partial_\alpha \langle s_1(\tau, -z), s_2(\tau, z) \rangle = \langle \nabla_{\partial_\alpha} s_1(\tau, -z), s_2(\tau, z) \rangle + \langle s_1(\tau, -z), \nabla_{\partial_\alpha} s_2(\tau, z) \rangle$$

and

$$z\frac{\partial}{\partial z} \langle s_1(\tau, -z), s_2(\tau, z) \rangle = \langle \nabla_{z\partial_z} s_1(\tau, -z), s_2(\tau, z) \rangle + \langle s_1(\tau, -z), \nabla_{z\partial_z} s_2(\tau, z) \rangle.$$

Remark 1.2. Flatness of the connection for the H^{even} directions is related to the associativity of big quantum product. Flatness for z and H^{even} together is related to the formula for

Remark 2.1. It is actually not so easy to corral the precise definition. Many sources have it in slightly different form. I am using Pandharipande's paper from 1998.

Let us now verify that this gives a flat section in the τ direction.

Lemma 2.2. We have for any β_1 and $\alpha \in H^{even}$

$$\nabla_{\partial_\alpha} S(\tau, z)\beta_1 = 0$$

as a function of (τ, z) with values in H^{even}.

Proof. The main idea behind the lemma is that the class of ψ_1 on $M_{0,n,\beta}$ is represented by a divisor which is the union of boundary components where the curve splits with labels 1 on one of the components and labels $2, 3$ on the other (and other labels anywhere they want to be).

I put more details in the Appendix. □

Unfortunately, we have $\nabla_{z\partial_z} S(\tau, z) \neq 0$. Rather, it needs to be adjusted a little bit to have it satisfy the connection fully, by multiplying it by an operator that is independent of τ.

Lemma 2.3. Let μ be as before and let ρ be the endomorphism of H^{even} which is the usual multiplication by $c_1(F)$. Then the function $S(\tau, z)z^{-\mu}z^{\rho}$ satisfies $\nabla_{z\partial_z} S(\tau, z)z^{-\mu}z^{\rho} = 0$.

Proof. Because the connection ∇ is flat, and $S(\tau, z)z^{-\mu}z^{\rho}$ is clearly horizontal in the τ direction, it suffices to check that $S(0, z)z^{-\mu}z^{\rho}$ satisfies

$$(z\frac{d}{dz} - \frac{1}{z}(c_1(F)\star_0) + \mu)S(0, z)z^{-\mu}z^{\rho} = 0.$$

This is a somewhat delicate calculation which I put in the Appendix. Initially, I could not get it work out, but Hiroshi Iritani kindly pointed out my error and fixed it. □

Proposition 2.4. The function $S(\tau, z)$ will be often restricted to $\tau = 0$. Then the fundamental solution satisfies an additional property

$$\lim_{z\to\infty} z^{\mu}S(0, z)z^{-\mu} = \mathrm{Id},$$

which determines it uniquely.

Proof. To verify this limit property, observe that we have for homogeneous β_1 and β_2 in $H^{2p_1}(X, \mathbb{C})$ and $H^{2p_2}(X, \mathbb{C})$ respectively,

$$\lim_{z\to\infty} \langle z^{\mu}S(0, z)z^{-\mu}\beta_1, \beta_2\rangle = \lim_{z\to\infty} \langle S(0, z)z^{-\mu}\beta_1, z^{-\mu}\beta_2\rangle$$

$$= \lim_{z\to\infty} \langle z^{-\mu}\beta_1, z^{-\mu}\beta_2\rangle + \lim_{z\to\infty} \sum_{0\neq d} z^{-p_1-p_2+\dim F} \int_{[M_{0,2,d}]_{virt}} \sum_{l\geq 0} z^{-l-1}(-1)^{l-1}\psi_1^l \cdot ev_1^*\beta_1 \cdot ev_2^*\beta_2$$

$$= \langle\beta_1, \beta_2\rangle + \lim_{z\to\infty} \sum_{\substack{0\neq d\in \mathrm{Eff} \\ (c_1(F)\cdot d)+\dim F-1-p_1-p_2\geq 0}} \int_{[M_{0,2,d}]_{virt}} z^{-(c_1(F)\cdot d)}(\pm 1)\psi_1^{(c_1(F)\cdot d)+\dim F-1-p_1-p_2} \cdot ev_1^*\beta_1 \cdot ev_2^*\beta_2.$$

While I am not certain how one can verify convergence,[2] at least formally we have that the limit is Id. Here is perhaps the first place where we explicitly use that F is Fano.

[2] I believe it follows from the fact that the power series satisfies the quantum differential equation.

For uniqueness, suppose that you have another fundamental solution of the form

$$S_1(z)z^{-\mu}z^{\rho},$$

with $\lim_{z\to\infty} z^{\mu}S_1(z)z^{-\mu} = \mathrm{Id}$. There exists a *constant* invertible matrix $C \in \mathrm{End}(H^{even})$ such that

$$S_1(z)z^{-\mu}z^{\rho} = S(0,z)z^{-\mu}z^{\rho}C.$$

We have $z^{\mu}S_1(z)z^{-\mu} = z^{\mu}S(0,z)z^{-\mu}z^{\rho}Cz^{-\rho}$. Therefore, we must have

$$\lim_{z\to\infty} z^{\rho}Cz^{-\rho} = \mathrm{Id}.$$

Note that $z^{\pm\rho}$ are polynomial in $\log z$, starting with identity. Therefore, on the left hand side we have C plus some $\mathrm{End}(H^{even})$-valued polynomial in $\log z$ with terms of positive degree. The only way this will have identity in the limit is if $C = \mathrm{Id}$. □

Remark 2.5. The previous proposition is very useful, because it may be often difficult to calculate $S(0,z)$ directly. However, the proposition allows us to find it, provided we can calculate the quantum differential equation.

3. Gamma conjecture. Proof for $\mathbb{CP}^n$.

To state Γ-conjecture I I need an additional conjecture called property $\mathcal{O}$.

Definition 3.1. A Fano variety F is said to satisfy property $\mathcal{O}$ if there exists a positive real number T such that the following statements hold.

- T is an eigenvalue of $c_1(F)\star_0 : H^{even} \to H^{even}$ on F of multiplicity 1.
- All other eigenvalues λ of $c_1(F)\star_0$ satisfy $|\lambda| \le T$.
- If for an eigenvalue λ there holds $|\lambda| = T$, then $\lambda = \xi T$ for $\xi^r = 1$ with r the Fano index of F.

Remark 3.2. We will see in a second that this assumption holds for $F = \mathbb{CP}^{n-1}$ where the eigenvalues are ξN with ξ running over all N-th roots of 1.

A consequence of property $\mathcal{O}$ (proved in Galkin-Golyshev-Iritani paper) is that among H^{even}-valued solutions of

$$\nabla_{z\partial_z} f(z) = 0$$

along $z \in \mathbb{R}_{>0}$ there is a unique up to scaling solution of $\nabla_{z\partial_z} f(z) = 0$ which has the smallest growth as $z \to 0+$. Specifically, this solution satisfies $\exp(\frac{T}{z})f(z) = O(z^{-m})$ for some m.

This solution can be given by

$$f(z) = S(0,z)z^{-\mu}z^{\rho}A$$

for some unique up to scaling class $A \in H^{even}$.

The Γ-conjecture I is the following statement.

Conjecture 3.3. (Γ conjecture I) Assuming F satisfies property $\mathcal{O}$, there holds

$$A = \hat{\Gamma}(F),$$

up to scaling.

Remark 3.4. It would be interesting to try to have a meaningful GW formula for $S(0,z)z^{-\mu}z^{\rho}\hat{\Gamma}(F)$, or $S(\tau,z)z^{-\mu}z^{\rho}\hat{\Gamma}(F)$, perhaps in terms of fat point markings.

4. Verifying Γ conjecture I for projective spaces.

Let $F = \mathbb{CP}^{N-1}$ be a projective space. Let us compute its quantum product at $\tau = 0$. We have

$$H^{even} = \oplus_{i=0}^{N-1} \mathbb{C} h^i$$

where h is the class of the hyperplane.

Lemma 4.1. There holds

$$h \star_0 h^i = \begin{cases} h^{i+1}, & 0 \le i \le N-2 \\ 1, & i = N-1. \end{cases}$$

Proof. We have

$$\langle h \star_0 h^i, h^j \rangle = \sum_d \int_{M_{0,3,d}(\mathbb{CP}^{N-1})} ev_1^* h \cdot ev_2^* h^i \cdot ev_3^* h^j.$$

The degree of this class is $1 + i + j \le 2N - 1$, the dimension of $M_{0,3,d}(\mathbb{CP}^{N-1})$ is

$$Nd + (N-1).$$

So the contributions occur only for $d = 0$ and $i + j = N - 2$ and $d = 1$ and $i = j = N - 1$. The first set of contributions acts just like a usual product (because the moduli in this case is $\mathbb{CP}^{N-1}$). The second type of contributions measures the number of lines ($d = 1$) in $\mathbb{CP}^{N-1}$ with three markings, so that the first marking is at a fixed hyperplane, and the other two are at fixed points, chosen generically. Clearly, there is one such line. This leads to $h \star_0 h^{N-1} = 1$. □

Therefore the action of $c_1(\mathbb{CP}^{N-1})\star_0 = Nh\star_0$ has eigenvalues ξN, so $\mathbb{CP}^{N-1}$ satisfies Conjecture $\mathcal{O}$.

Let us now calculate the quantum connection and write its fundamental solution $S(0,z)z^{-\mu}z^{\rho}$. For simplicity, let us consider a solution of the form

$$f(z) = \sum_{i=0}^{N-1} f_i(z) z^{i-(N-1)/2} h^{N-1-i}.$$

We have

$$\nabla_{z\partial_z} f(z) = \sum_{i=0}^{N-1} (z\frac{\partial}{\partial z} f_i(z)) z^{i-(N-1)/2} h^{N-1-i} + \sum_{i=0}^{N-1} f_i(z)(i-(N-1)/2) z^{i-(N-1)/2} h^{N-1-i}$$

$$-\frac{N}{z}\sum_{i=1}^{N-1} f_i(z) z^{i-(N-1)/2} h^{N-i} - \frac{N}{z} f_0(z) z^{-(N-1)/2} h^0 + \sum_{i=0}^{N-1} f_i(z) z^{i-(N-1)/2}(N-1-i-(N-1)/2) h^{N-1-i}$$

$$= \sum_{i=0}^{N-1} (z\frac{\partial}{\partial z} f_i(z)) z^{i-(N-1)/2} h^{N-1-i} - \frac{N}{z}\sum_{i=1}^{N-1} f_i(z) z^{i-(N-1)/2} h^{N-i} - \frac{N}{z} f_0(z) z^{-(N-1)/2} h^0.$$

By comparing coefficients at h^{N-1-i} we see that $\nabla_{z\partial_z} f(z) = 0$ is equivalent to

$$(z\frac{\partial}{\partial z} f_i(z)) = \begin{cases} N f_{i+1}(z), & i = 0, \dots, N-2 \\ N z^{-N} f_0(z), & i = N-1. \end{cases}$$

In other words, $f_0(z)$ satisfies the differential equation

$$(z\frac{\partial}{\partial z})^N f_0(z) - N^N z^{-N} f_0(z) = 0, \tag{4.1}$$

and the other f_i are its logarithmic derivatives (times N^{-i}).

We can now calculate the fundamental solution $S(0,z)z^{-\mu}z^{\rho}$. Define the function $\Pi(z)$ with values in $(H^{even})^\vee = (\mathbb{C}[h]/\langle h^N\rangle)^\vee$

$$\Pi(z)(\alpha) = \int_{\mathbb{CP}^{N-1}} \sum_{n=0}^{\infty} \frac{\alpha\, z^{-N(n-h)}}{\big((h-n)(h-n+1)\cdots(h-1)\big)^N}$$

where for $n=0$ the empty product is 1. The meaning of z^{Nh} is, of course $\exp(Nh\log z)$ and we are working away from $z \in \mathbb{R}_{\leq 0}$. Convergence is clear. This function is immediately seen to satisfy

$$(z\frac{\partial}{\partial z})^N \Pi(z) = N^N z^{-N}\Pi(z).$$

For each α it then gives a solution of quantum differential equation

$$\begin{aligned}\Phi(z,\alpha) &= \sum_{i=0}^{N-1} z^{i-(N-1)/2} h^{N-1-i}\Big(\frac{z}{N}\frac{d}{dz}\Big)^i \int_{\mathbb{CP}^{N-1}} \sum_{n=0}^{\infty} \frac{\alpha\, z^{-N(n-h)}}{\big((h-n)(h-n+1)\cdots(h-1)\big)^N} \\ &= \sum_{i=0}^{N-1} z^{i-(N-1)/2} h^{N-1-i} \int_{\mathbb{CP}^{N-1}} \sum_{n=0}^{\infty} \frac{(h-n)^i\alpha\, z^{-N(n-h)}}{\big((h-n)(h-n+1)\cdots(h-1)\big)^N}.\end{aligned}$$

Proposition 4.2. The fundamental solution $S(0,z)z^{-\mu}z^{\rho}$ is given by $\Phi(z)$.

Proof. Because $c_1(\mathbb{CP}^{N-1}) = Nh$, we get

$$z^{\mu}\Phi(z, z^{-\rho}\alpha) = \sum_{i=0}^{N-1} h^{N-1-i} \int_{\mathbb{CP}^{N-1}} \sum_{n=0}^{\infty} \frac{(h-n)^i\alpha\, z^{-Nn}}{\big((h-n)(h-n+1)\cdots(h-1)\big)^N}.$$

In the limit as $z\to\infty$ only the $n=0$ terms survive and we get

$$\alpha \mapsto \sum_{i=0}^{N-1} h^{N-1-i}\int_{\mathbb{CP}^{N-1}} h^i\alpha = \alpha.$$

Thus Φ is the fundamental solution. □

We will now the smallest solution as $z\to 0+$. Let us define

$$\Psi(z) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s)^N z^{Ns}\, ds$$

where c is some (any) positive real number. We use the Stirling approximation of the Γ function

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right)$$

which is valid in any sector $\mathrm{Arg}(z) \in (-\pi+\epsilon, \pi+\epsilon)$, to conclude that

$$\begin{aligned}|\Gamma(s)z^s| &\sim |s|^{-\frac{1}{2}}\exp(\mathrm{Re}(s\log(s) - s + s\log z) \\ &= |s|^{-\frac{1}{2}}\exp(c\mathrm{Re}(\log(s)) - \mathrm{Im}(s)\mathrm{Im}(\log(s)) - c + c\log|z| - \mathrm{Im}(s)\mathrm{Arg}(z)).\end{aligned}$$

As $s \to c \pm \mathrm{i}\infty$, we see that $\mathrm{Im}(\log(s)) \to \pm\frac{\pi}{2}$, so the fastest growing term inside exponential is about $-(\frac{\pi}{2} \pm \mathrm{Arg}(z))|\mathrm{Im}(s)|$. The other terms there are logarithmic or constant in $|\mathrm{Im}(s)|$, so convergence is clear.

It is clear that $\Psi(z)$ is independent from the choice of c, since the poles of Γ are at $\mathbb{Z}_{\leq 0}$. It is clear that as one applies $(\frac{z}{N}\frac{d}{dz})^N$ to this integral, we get extra s^N inside. This amounts to passing from c to $c+1$. Thus Ψ satisfies (4.1).

We observe that $\Psi(z) = \Pi(z)\hat{\Gamma}(\mathbb{CP}^{N-1})$ (which means that the corresponding H^{even}-valued function is $\Phi(z)\hat{\Gamma}(\mathbb{CP}^{N-1})$).

To do so, observe that $\Psi(z)$ is equal to the sum of residues of the integrand at $s = 0, -1, -2, \ldots$. Indeed, the remainder term is estimated as

$$\frac{1}{2\pi \mathrm{i}}\int_{-k+\frac{1}{2}+\mathrm{i}\mathbb{R}} \Gamma(s)^N z^{Ns}\,ds = \frac{1}{2\pi \mathrm{i}}\int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \Gamma(s-k)^N z^{N(s-k)}\,ds = \frac{1}{2\pi \mathrm{i} z^{kN}}\int_{\frac{1}{2}+\mathrm{i}\mathbb{R}} \Gamma(s)^N z^{Ns} \prod_{j=1}^{k}(s-j)^{-N}\,ds$$

The integrand is dominated by the original one, so we see that the sum of residues converges to $\Psi(z)$. We now have

$$\Psi(z) = \sum_{n=0}^{\infty} \mathrm{Res}_{s=-n}\Big(\Gamma(s)^N z^{Ns}\Big) = \sum_{n=0}^{\infty} \mathrm{Res}_{s=0}\Big(\Gamma(s-n)^N z^{N(s-n)}\Big)$$

$$= \sum_{n=0}^{\infty} \mathrm{Res}_{s=0}\Big(\frac{\Gamma(1+s)^N z^{N(s-n)}}{s^N (s-1)^N \cdots (s-n)^N}\Big) = \sum_{n=0}^{\infty} \int_{\mathbb{CP}^{N-1}} \Big(\frac{\Gamma(1+h)^N z^{N(h-n)}}{(h-1)^N \cdots (h-n)^N}\Big)$$

$$= \Pi(z)(\Gamma(1+h)^N).$$

It remains to observe that since $c(\mathbb{CP}^{N-1}) = (1+h)^N$, we have $\hat{\Gamma}(\mathbb{CP}^{N-1}) = \Gamma(1+h)^N$.

Galkin-Golyshev-Iritani paper claims that asymptotically

$$\Psi(z) = C z^{(N-1)/2} e^{-\frac{N}{z}}(1 + O(z))$$

as $z \to 0+$ with a reference to Meijer (1946). Meijer refers to Barnes (1906), which is hard to make sense of, although it is available online. There is a more useable, although still much more general reference "Asymptotic expansions and analytic continuations for a class of Barnes-integrals" by Braaksma from 1963. I will sketch the main points of the argument. One has (after some justification)

$$\frac{1}{2\pi \mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}} \Gamma(s+\alpha) z^s\,ds = \frac{z^{-\alpha}}{2\pi \mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}} \Gamma(s) z^s\,ds \tag{4.2}$$

$$= z^{-\alpha}\sum_{j=0}^{\infty} \mathrm{Res}_{s=-j}\Gamma(s) z^s = z^{-\alpha}\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^{-j} = z^{-\alpha}\, e^{-\frac{1}{z}}.$$

From Stirling formula, one has

$$\Gamma(s)^N = C\,N^{-Ns}\,\Gamma(Ns - \frac{1}{2}(N-1))(1 + O(s^{-1}))$$

for some (explicit) constant C. Then one has

$$\Psi(z) = \frac{C}{2\pi \mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}} \Gamma(Ns - \frac{1}{2}(N-1))\,(z/N)^{Ns}\,ds + \frac{C}{2\pi \mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}} \Gamma(Ns - \frac{1}{2}(N-1)) O(s^{-1})\,(z/N)^{Ns}\,ds.$$

By (4.2), the first term is equal to $C z^{\frac{1}{2}(N-1)} e^{-\frac{N}{z}}$. The second term is shown to be of lower asymptotics (this is the most technical part, but it is not too bad in Braaksma's paper).

In fact, Braaksma's paper (and others) cover a far larger range of integrals with products of Γ functions in it. It is likely that if one were to prove Γ conjecture no new tools would be needed at this step.

Remark 4.3. One can also look at

$$\frac{1}{2\pi \mathrm{i}}\int_{c+\mathrm{i}\mathbb{R}} \Gamma(s)^N z^{Ns} e^{2\pi \mathrm{i} ks}\, ds$$

which correspond to solutions of $\Phi(z)\hat{\Gamma}(\mathbb{CP}^{N-1})\mathrm{Ch}(\mathcal{O}(k))$ where Ch is the weighted Chern character. However, the paper is interested in sections that are asymptotically of the form $e^{-N\xi/z}$ for ξ a root of 1 along a sector near $z \to 0+$. They prove that these sections correspond to mutations of the standard Beilinson exceptional collection, but do not calculate them explicitly.

5. Appendix: verifying flatness of quantum connection

In this section we will verify the flatness of

$$\nabla_{z\partial_z} := z\frac{\partial}{\partial z} - \frac{1}{z}(E\star_\tau) + \mu, \qquad \nabla_{\partial_\alpha} := \frac{\partial}{\partial \alpha} + \frac{1}{z}(\alpha\star_\tau).$$

There is probably a more elegant way of doing it, but for the sake of my sanity I want to do it via brute force.

Let us first verify that for any α and β in H^{even}

$$\nabla_{\partial_\alpha}\nabla_{\partial_\beta} s(\tau, z) - \nabla_{\partial_\beta}\nabla_{\partial_\alpha} s(\tau, z) = 0$$

for any H^{even}-valued holomorphic s. We have

$$(\nabla_{\partial_\alpha}\nabla_{\partial_\beta} - \nabla_{\partial_\beta}\nabla_{\partial_\alpha})s(\tau, z)$$

$$= [\frac{\partial}{\partial_\alpha}, \frac{\partial}{\partial_\beta}]s + \frac{1}{z}\Big([\frac{\partial}{\partial_\alpha}, \beta\star_\tau] - [\frac{\partial}{\partial_\beta}, \alpha\star_\tau]\Big)s(\tau, z) + \frac{1}{z^2}[\alpha\star_\tau, \beta\star_\tau]s$$

(the vector fields commute, and multiplication maps commute due to associativity and commutativity)

$$= \frac{1}{z}\Big([\frac{\partial}{\partial_\alpha}, \beta\star_\tau] - [\frac{\partial}{\partial_\beta}, \alpha\star_\tau]\Big)s(\tau, z). \tag{5.1}$$

To compute $[\frac{\partial}{\partial_\alpha}, \beta\star_\tau]s$ we will compute

$$\langle [\frac{\partial}{\partial_\alpha}, \beta\star_\tau]s(\tau, z), u\rangle$$

for arbitrary constant $u \in H^{even}$. We have

$$\langle \beta \star_\tau s(\tau, z), u\rangle = \sum_{d,n} \frac{1}{n!}\langle \beta, s(\tau, z), u, \tau, \ldots, \tau\rangle_{0,3+n,d}$$

so

$$\frac{\partial}{\partial_\alpha}\langle \beta \star_\tau s(\tau, z), u\rangle = \langle \beta \star_\tau (\frac{\partial}{\partial_\alpha} s(\tau, z)), u\rangle + \sum_{d,n} \frac{n}{n!}\langle \beta, s(\tau, z), u, \alpha, \tau, \ldots, \tau\rangle_{0,3+n,d}.$$

The point is that we need to differentiate each $ev_i^*\tau$ in the α direction, i.e. to see the rate of change as τ is replaced by $\tau + \epsilon\alpha$. There are n occurrences of τ and each of them gets a term. We also use the obvious fact that GW invariants are independent of the order of the markings. This leads to

$$\langle [\frac{\partial}{\partial_\alpha}, \beta\star_\tau] s(\tau,z), u\rangle = \sum_{d,n} \frac{1}{n!} \langle \beta, s(\tau,z), u, \alpha, \tau, \ldots, \tau\rangle_{0,4+n,d}$$

which is clearly symmetric in α and β, This shows that (5.1) is zero, as desired.

We will now verify that

$$(\nabla_{\partial_\alpha}\nabla_{z\partial_z} - \nabla_{z\partial_z}\nabla_{\partial_\alpha}) s(\tau,z) = 0.$$

After observing that the coordinate vector fields on the base commute, that operator μ doesn't depend on τ, and that $\alpha\star_\tau$ and $E\star_\tau$ commute, we have

$$\begin{aligned}
[\nabla_{\partial_\alpha}, \nabla_{z\partial_z}] &= [\frac{\partial}{\partial\alpha} + \frac{1}{z}(\alpha\star_\tau), z\frac{\partial}{\partial z} - \frac{1}{z}(E\star_\tau) + \mu] \\
&= -\frac{1}{z}[\frac{\partial}{\partial\alpha}, E\star_\tau] + [\frac{1}{z}, z\frac{\partial}{\partial z}]\alpha\star_\tau + \frac{1}{z}[\alpha\star_\tau, \mu] \\
&= \frac{1}{z}\Big(-[\frac{\partial}{\partial\alpha}, E\star_\tau] + \alpha\star_\tau + [\alpha\star_\tau, \mu]\Big). \qquad (5.2)
\end{aligned}$$

As before, we will apply the operator in the parenthesis to an H^{even}-valued function $s(\tau,z)$ and pair it with $u \in H^{even}$. We will take u to be homogeneous, i.e. $u \in H^{2\deg(u)}(F,\mathbb{C})$. We will also have s homogeneous in $H^{2\deg(s)}(F,\mathbb{C})$.

Let $\{\phi_i\}$ be a homogeneous basis of H^{even}, with $\phi_i \in H^{2p_i}(F,\mathbb{C})$. Let $\tau = \sum_i \tau_i\phi_i$. Without loss of generality we may further assume that $\alpha = \phi_1$. Our definition of E implies that

$$E = E(\tau) = c_1(F) + \sum_i (1-p_i)\tau_i\phi_i.$$

Therefore, when we differentiate in the direction of $\alpha = \phi_1$ we have $\frac{\partial}{\partial\alpha}E = (1-p_1)\phi_1$. Then as before, we get

$$\langle [\frac{\partial}{\partial\alpha}, E\star_\tau] s(\tau,z), u\rangle = \langle (1-p_1)\phi_1 \star_\tau s, u\rangle + \sum_{d,n}\frac{1}{n!}\langle E, s(\tau,z), u, \phi_1, \tau, \ldots, \tau\rangle_{0,4+n,d}$$

Since $\mu\, s = (\deg s - \frac{1}{2}\dim F)s$ and μ is skew for the pairing, we get

$$\begin{aligned}
\langle [\alpha\star_\tau, \mu] s, u\rangle &= (\deg s - \frac{1}{2}\dim F)\langle \phi_1 \star_\tau s, u\rangle - \langle \mu\phi_1 \star_\tau s, u\rangle \\
&= (\deg s - \frac{1}{2}\dim F)\langle \phi_1 \star_\tau s, u\rangle + \langle \phi_1 \star_\tau s, \mu u\rangle \\
&= (\deg s + \deg u - \dim F)\langle \phi_1 \star_\tau s, u\rangle.
\end{aligned}$$

To continue with (5.2), we get

$$\begin{aligned}
&\langle \Big(-[\frac{\partial}{\partial\alpha}, E\star_\tau] + \alpha\star_\tau + [\alpha\star_\tau, \mu]\Big) s, u\rangle \\
&= -\langle (1-p_1)\phi_1 \star_\tau s, u\rangle - \sum_{d,n}\frac{1}{n!}\langle E, s(\tau,z), u, \phi_1, \tau, \ldots, \tau\rangle_{0,4+n,d} \\
&\quad + \langle \phi_1 \star_\tau s, u\rangle + (\deg s + \deg u - \dim F)\langle \phi_1 \star_\tau s, u\rangle
\end{aligned}$$

$$(5.3)\qquad = (p_1+\deg s+\deg u-\dim F)\langle \phi_1\star_\tau s,u\rangle - \sum_{d,n}\frac{1}{n!}\langle E, s(\tau,z),u,\phi_1,\tau,\dots,\tau\rangle_{0,4+n,d}.$$

Let us simplify the GW invariants with E. First of all, the divisor axiom gives terms

$$(5.4)\qquad -\sum_{d,n}\frac{1}{n!}(c_1(F)\cdot d)\,\langle s,u,\phi_1,\tau,\dots,\tau\rangle_{0,3+n,d}.$$

We also get the term for

$$(5.5)\qquad -\sum_{i=1}^{\dim H^{even}}\sum_{d,n}\frac{1}{n!}(1-p_i)\tau_i\langle s,u,\phi_1,\phi_i,\tau,\dots,\tau\rangle_{0,4+n,d}.$$

Let us expand these terms for $\tau=\sum_i\tau_i\phi_i$ and calculate the coefficients at

$$\langle s,u,\phi_1,...,\phi_1,\dots,\dots\rangle_{0,2+\sum_i k_i,d}$$

where each ϕ_i occurs k_i times. Note that $k_1\geq 1$, and the other k_i are nonnegative. The term (5.4) contributes (using multinomial theorem)

$$-(c_1(F)\cdot d)\frac{\tau_1^{k_1-1}}{(k_1-1)!}\prod_{i>1}\frac{\tau_i^{k_i}}{k_i!}.$$

The term (5.5) contributes (using convention $(-1)!=\infty$)

$$-\sum_{j>1}(1-p_j)\tau_j\frac{\tau_1^{k_1-1}}{(k_1-1)!}\frac{\tau_j^{k_j-1}}{(k_j-1)!}\prod_{i\neq 1,j}\frac{\tau_i^{k_i}}{k_i!}-(1-p_1)\tau_1\frac{\tau_1^{k_1-2}}{(k_1-2)!}\prod_{i>1}\frac{\tau_i^{k_i}}{k_i!}.$$

Together, we get

$$\begin{aligned}&\frac{\tau_1^{k_1-1}}{(k_1-1)!}\prod_{i>1}\frac{\tau_i^{k_i}}{k_i!}\Big(-(c_1(F)\cdot d)-\sum_{j>1}(1-p_j)k_j-(1-p_1)(k_i-1)\Big)\\ &=\frac{\tau_1^{k_1-1}}{(k_1-1)!}\prod_{i>1}\frac{\tau_i^{k_i}}{k_i!}\Big(-(c_1(F)\cdot d)+\sum_j p_jk_j-\sum_j k_j+(1-p_1)\Big)\end{aligned}$$

If we go back to (5.3), we observe that the coefficient by

$$\langle s,u,\phi_1,...,\phi_1,\dots,\dots\rangle_{0,2+\sum_i k_i,d}$$

is given by

$$\begin{aligned}&\frac{\tau_1^{k_1-1}}{(k_1-1)!}\prod_{i>1}\frac{\tau_i^{k_i}}{k_i!}\Big(\deg s+\deg u-\dim F+p_1-(c_1(F)\cdot d)+\sum_j p_jk_j-\sum_j k_j+(1-p_1)\Big)\\ &=\frac{\tau_1^{k_1-1}}{(k_1-1)!}\prod_{i>1}\frac{\tau_i^{k_i}}{k_i!}\Big(\deg s+\deg u+\sum_j p_jk_j-\dim F-(c_1(F)\cdot d)-\sum_j k_j+1\Big).\end{aligned}$$

It remains to observe that the GW invariant is zero unless the sum of the degrees of the variables is equal to the virtual dimension of the moduli space, i.e.

$$\deg s+\deg u+\sum_j p_jk_j=(c_1(F)\cdot d)+\sharp(points)+\dim F-3$$

$$= (c_1(F) \cdot d) + \sum_j k_j + \dim F - 1.$$

So in the above expression either the GW invariant or the coefficient are zero.

This finishes the proof of flatness of the connection.

6. Appendix: quantum connection and Poincare pairing

In this section we will verify that the quantum connection is compatible with the Poincare pairing in the sense of

$$\partial_\alpha \langle s_1(\tau,-z), s_2(\tau,z)\rangle = \langle \nabla_{\partial_\alpha} s_1(\tau,-z), s_2(\tau,z)\rangle + \langle s_1(\tau,-z), \nabla_{\partial_\alpha} s_2(\tau,z)\rangle$$

and

$$z\frac{\partial}{\partial z}\langle s_1(\tau,-z), s_2(\tau,z)\rangle = \langle \nabla_{z\partial_z} s_1(\tau,-z), s_2(\tau,z)\rangle + \langle s_1(\tau,-z), \nabla_{z\partial_z} s_2(\tau,z)\rangle.$$

This is a straightforward calculation.

For the first term of the right hand side of the first equation, we get

$$\langle \nabla_{\partial_\alpha} s_1(\tau,-z), s_2(\tau,z)\rangle = \langle \partial_\alpha s_1(\tau,-z), s_2(\tau,z)\rangle + \frac{1}{(-z)}\langle \alpha \star_\tau s_1(\tau,-z), s_2(\tau,z)\rangle.$$

Similarly,

$$\langle s_1(\tau,-z), \nabla_{\partial_\alpha} s_2(\tau,z)\rangle = \langle s_1(\tau,-z), \partial_\alpha s_2(\tau,z)\rangle + \frac{1}{z}\langle s_1(\tau,-z), \alpha \star_\tau s_2(\tau,z)\rangle.$$

As you add these, the $\frac{1}{z}$ terms cancel from the definition of the quantum product, and the desired equality follows.

For the first term on the right hand side of the second equation we get

$$\langle \nabla_{z\partial_z} s_1(\tau,-z), s_2(\tau,z)\rangle = \langle z\frac{\partial}{\partial z} s_1(\tau,-z), s_2(\tau,z)\rangle + \frac{1}{(-z)}\langle E\star_\tau s_1(\tau,-z), s_2(\tau,z)\rangle + \langle \mu s_1(\tau,-z), s_2(\tau,z)\rangle.$$

For the second term we get

$$\langle s_1(\tau,-z), \nabla_{z\partial_z} s_2(\tau,z)\rangle = \langle s_1(\tau,-z), z\frac{\partial}{\partial z} s_2(\tau,z)\rangle + \frac{1}{z}\langle s_1(\tau,-z), E\star_\tau s_2(\tau,z)\rangle + \langle s_1(\tau,-z), \mu s_2(\tau,z)\rangle.$$

When you add them, the $\frac{1}{z}$ terms cancel from the definition of the quantum product and the μ terms cancel from skew-symmetry of μ.

7. Appendix: proof of Lemma 2.2

We will evaluate $\langle \nabla_{\partial_\alpha} S(\tau,z)(\beta_1), \beta_2\rangle$ for a fixed β_2. We will also pick dual bases γ_k and γ^k of H^{even}. Then we have

$$\langle \nabla_{\partial_\alpha} S(\tau,z)\beta_1, \beta_2\rangle$$

$$= \frac{1}{z}\langle \alpha \star_\tau \beta_1, \beta_2\rangle + \sum_{n\geq 0, d\in \mathrm{Eff}} \sum_k \frac{\langle \gamma^k, \beta_2\rangle}{n!} \int_{[M_{0,n+3,d}]^{virt}} \frac{ev_1^*\beta_1}{-z-\psi_1} \cdot ev_2^*\gamma_k \cdot ev_3^*\alpha \cdot \prod_{i=4}^{n+1} ev_i^*\tau$$

$$+ \frac{1}{z} \sum_{m\geq 0, d'\in \mathrm{Eff}} \sum_{\substack{n\geq 0, d\in \mathrm{Eff} \\ (n,d)\neq (0,0)}} \sum_k \frac{1}{n!m!} \int_{[M_{0,m+3,d'}]^{virt}} ev_1^*\alpha \cdot ev_2^*\beta_2 \cdot ev_3^*\gamma^k \cdot \prod_{i=4}^{m+3} ev_i^*\tau$$

$$\int_{[M_{0,n+2,d}]_{virt}} \frac{ev_1^*\beta_1}{-z-\psi_1} \cdot ev_2^*\gamma_k \cdot \prod_{i=3}^{n+2} ev_i^*\tau$$

$$= \frac{1}{z}\langle \alpha \star_\tau \beta_1, \beta_2\rangle + \sum_{n\geq 0, d\in \mathrm{Eff}} \frac{1}{n!}\int_{[M_{0,n+3,d}]_{virt}} \frac{ev_1^*\beta_1}{-z-\psi_1} \cdot ev_2^*\beta_2 \cdot ev_3^*\alpha \cdot \prod_{i=4}^{n+1} ev_i^*\tau$$

$$+\frac{1}{z}\sum_{l\geq 0, d''\in \mathrm{Eff}} \frac{1}{l!}\int_{[M_{0,l+3,d''}]_{virt}} \frac{\psi_1 \cdot ev_1^*\beta_1}{-z-\psi_1} \cdot ev_2^*\alpha \cdot ev_3^*\beta_2 \cdot \prod_{i=4}^{l+3} ev_i^*\tau$$

$$= \sum_{l\geq 0, d''\in \mathrm{Eff}} \frac{1}{l!}\int_{[M_{0,l+3,d''}]_{virt}} \left(\frac{1}{z} + \frac{1}{-z-\psi_1} + \frac{1}{z}\frac{\psi_1}{(-z-\psi_1)}\right) \cdot ev_1^*\beta_1 \cdot ev_2^*\alpha \cdot ev_3^*\beta_2 \cdot \prod_{i=4}^{l+3} ev_i^*\tau = 0.$$

Here we have used the property that ψ_1 class on $M_{0,l+3,d''}$ can be written as sum of divisors of the loci of curves where markings 1 and $2,3$ are located at different components.

8. Appendix: proving that $S(0,z)z^{-\mu}z^{\rho}$ is a flat section.

We will first formulate the following lemma, whose statement and proof I owe to Hiroshi Iritani.

Lemma 8.1. For $d \neq 0$ and $l \geq 1$ there holds

$$\int_{[M_{0,3,d}]_{virt}} \psi_1^l ev_1^*\beta_1 \cdot ev_2^* c_1(F) \cdot ev_3^*\beta_2 = (c_1(F)\cdot d)\int_{[M_{0,2,d}]_{virt}} \psi_1^l ev_1^*\beta_1 \cdot ev_2^*\beta_2$$

$$+\int_{[M_{0,2,d}]_{virt}} \psi_1^{l-1} ev_1^*(c_1(F)\beta_1) \cdot ev_2^*\beta_2.$$

Proof. For ψ_1 classes on $M_{0,3,d}$ and $M_{0,2,d}$ with the map $\pi : M_{0,3,d} \to M_{0,2,d}$ that forgets the second marking there holds

$$\psi_1 = \pi^*\psi_1 + D$$

where D is the divisor of curves where first and second markings are on a curve of degree zero and the last marking is on the other component(s). We also have $\psi_1 \cdot D = 0$ from the description of ψ_1. This implies for $l \geq 1$

$$\psi_1^l = \psi_1(\pi^*\psi_1 + D)^{l-1} = \psi_1\pi^*\psi_1^{l-1} = \pi^*\psi_1^l + D\pi^*\psi_1^{l-1}.$$

The contribution of the first term leads to $(c_1(F)\cdot d)\int_{[M_{0,2,d}]_{virt}} \psi_1^l ev_1^*\beta_1 \cdot ev_2^*\beta_2$. The contribution of the second term is the second term of the above sum since working on D makes us multiply the condition on the markings. ∎

Corollary 8.2.

$$\int_{[M_{0,3,d}]_{virt}} \frac{ev_1^*\beta_1}{-z-\psi_1} \cdot ev_2^* c_1(F) \cdot ev_3^*\beta_2 = (c_1(F)\cdot d)\int_{[M_{0,2,d}]_{virt}} \frac{ev_1^*\beta_1}{-z-\psi_1} \cdot ev_2^*\beta_2$$

$$-\frac{1}{z}\int_{[M_{0,2,d}]_{virt}} \frac{ev_1^*(c_1(F)\beta_1)}{-z-\psi_1} \cdot ev_2^*\beta_2.$$

Now let me get back to the proof of Lemma 2.2. Recall that

$$S(0,z)z^{-\mu}z^{\rho}\beta_1 = z^{-\mu}z^{\rho}\beta_1 + \sum_{0\neq d\in \mathrm{Eff}}\sum_k \gamma^k \int_{[M_{0,2,d}]_{virt}} \frac{ev_1^*(z^{-\mu}z^{\rho}\beta_1)}{-z-\psi_1}\cdot ev_2^*\gamma_k$$

$$= z^{-\mu}z^{\rho}\beta_1 + \sum_{0\neq d\in \mathrm{Eff}}\sum_{l\geq 0}\sum_k \gamma^k \int_{[M_{0,2,d}]_{virt}} ev_1^*(z^{-\mu-l-1}z^{\rho}\beta_1)\cdot(-1)^{l-1}\psi^l\cdot ev_2^*\gamma_k.$$

We also have

$$(z\frac{d}{dz})z^{-\mu-l-1}z^{\rho} = (-\mu-l-1)z^{-\mu-l-1}z^{\rho} + z^{-\mu-l-1}\rho z^{\rho} = (\rho z^{-1}-\mu-l-1)z^{-\mu-l-1}z^{\rho}$$

which implies

$$\langle\Big(z\frac{d}{dz}+\mu\Big)S(0,z)z^{-\mu}z^{\rho}\beta_1,\beta_2\rangle = \langle\Big(z\frac{d}{dz}\Big)S(0,z)z^{-\mu}z^{\rho}\beta_1,\beta_2\rangle - \langle S(0,z)z^{-\mu}z^{\rho}\beta_1,\mu\beta_2\rangle$$

$$= \langle c_1(F)\, z^{-\mu-1}z^{\rho}\beta_1,\beta_2\rangle + \sum_{0\neq d\in \mathrm{Eff}}\int_{[M_{0,2,d}]_{virt}}\sum_{l\geq 0} ev_1^*((\rho z^{-1}-\mu-l-1)z^{-\mu-l-1}z^{\rho}\beta_1)\cdot(-1)^{l-1}\psi_1^l\cdot ev_2^*\beta_2$$

$$-\sum_{0\neq d\in \mathrm{Eff}}\int_{[M_{0,2,d}]_{virt}}\sum_{l\geq 0} ev_1^*(z^{-\mu-l-1}z^{\rho}\beta_1)\cdot(-1)^{l-1}\psi_1^l\cdot ev_2^*\mu\beta_2.$$

Since virtual dimension of $M_{0,2,d}$ is $(c_1(F)\cdot d)+\dim F-1$, and degree of ψ_1 is 1, we get

$$\int_{[M_{0,2,d}]_{virt}} \psi_1^l\cdot ev_1^*x\cdot ev_2^*\mu y = \int_{[M_{0,2,d}]_{virt}} \psi_1^l\cdot ev_1^*(c_1(F)\cdot d-\mu-l-1)x\cdot ev_2^* y.$$

This allows us to simplify:

$$\langle\Big(z\frac{d}{dz}+\mu\Big)S(0,z)z^{-\mu}z^{\rho}\beta_1,\beta_2\rangle = \langle c_1(F)\, z^{-\mu-1}z^{\rho}\beta_1,\beta_2\rangle$$

$$+\sum_{0\neq d\in \mathrm{Eff}}\int_{[M_{0,2,d}]_{virt}}\sum_{l\geq 0} ev_1^*((\rho z^{-1}-c_1(F)\cdot d)z^{-\mu-l-1}z^{\rho}\beta_1)\cdot(-1)^{l-1}\psi_1^l\cdot ev_2^*\beta_2$$

$$= \langle c_1(F)\, z^{-\mu-1}z^{\rho}\beta_1,\beta_2\rangle + \sum_{0\neq d\in \mathrm{Eff}}\int_{[M_{0,2,d}]_{virt}} \frac{ev_1^*((\rho z^{-1}-c_1(F)\cdot d)z^{-\mu}z^{\rho}\beta_1)}{-z-\psi_1}\cdot ev_2^*\beta_2. \tag{8.1}$$

On the other hand,

$$\langle\frac{1}{z}c_1(F)\star_0 S(0,z)z^{-\mu}z^{\rho}\beta_1,\beta_2\rangle = \langle c_1(F)\star_0\ z^{-\mu-1}z^{\rho}\beta_1,\beta_2\rangle$$

$$+\frac{1}{z}\sum_{d\neq 0}\sum_{d_1}\sum_k \int_{[M_{0,3,d_1}]_{virt}} ev_1^*c_1(F)\cdot ev_2^*\gamma^k\cdot ev_3^*\beta_2\int_{[M_{0,2,d}]_{virt}}\frac{ev_1^*(z^{-\mu}z^{\rho}\beta_1)}{-z-\psi_1}\cdot ev_2^*\gamma_k$$

$$= \langle c_1(F)z^{-\mu-1}z^{\rho}\beta_1,\beta_2\rangle + \frac{1}{z}\sum_{d\neq 0}\int_{[M_{0,3,d}]_{virt}} ev_1^*z^{-\mu}z^{\rho}\beta_1\cdot ev_2^*c_1(F)\cdot ev_3^*\beta_2$$

$$+\frac{1}{z}\sum_{d\neq 0}\int_{[M_{0,3,d}]_{virt}}\frac{\psi_1 ev_1^*z^{-\mu}z^{\rho}\beta_1}{-z-\psi_1}\cdot ev_2^*c_1(F)\cdot ev_3^*\beta_2$$

$$= \langle c_1(F)z^{-\mu-1}z^{\rho}\beta_1,\beta_2\rangle - \sum_{d\neq 0}\int_{[M_{0,3,d}]_{virt}}\frac{ev_1^*z^{-\mu}z^{\rho}\beta_1}{-z-\psi_1}\cdot ev_2^*c_1(F)\cdot ev_3^*\beta_2.$$

We now use Corollary 8.2 to get

$$\langle \frac{1}{z} c_1(F) \star_0 S(0,z) z^{-\mu} z^{\rho} \beta_1, \beta_2 \rangle = \langle c_1(F) z^{-\mu-1} z^{\rho} \beta_1, \beta_2 \rangle$$

$$- \sum_{d \neq 0} \int_{[M_{0,2,d}]_{virt}} (c_1(F) \cdot d) \frac{ev_1^*(z^{-\mu} z^{\rho} \beta_1)}{-z-\psi_1} \cdot ev_2^* \beta_2 + \frac{1}{z} \sum_{d \neq 0} \int_{[M_{0,2,d}]_{virt}} \frac{ev_1^*(c_1(F) z^{-\mu} z^{\rho} \beta_1)}{-z-\psi_1} \cdot ev_2^* \beta_2$$

which matches (8.1) and finishes the argument.