HMS for K3's reading group:

10/12/2016

F. Lin, Torelli thm for K3s

1) Basics
2) Torelli theorems
3) Moduli spaces, period map

§1. Let X be a cplx. surface. Say X is K3 if $H^1(X, \mathcal{O}_X) = 0$ and $K_X \cong \mathcal{O}_X$ (canon. bundle trivial).

Facts: a) X K3 $\Rightarrow$ X is Kähler, (but not necessarily projective).

b) All K3s are diffeomorphic (will follow from description of moduli space.).

Examples: (i)
- quartic $\subset \mathbb{P}^3$
- (2,3) complete intersection in $\mathbb{P}^4$
- (2,2,2) " " in $\mathbb{P}^5$.

give projective K3's.

(ii) Kummer surfaces: Say T a complex torus, meaning $\mathbb{C}^2/\Gamma$ (lattice).

The action $\{z \mapsto -z\}$ has 16 fixed points on T, the 2-torsion points on T.

$\Rightarrow T/_{\{z\mapsto -z\}}$ has 16 nodal singularities. (each of type A_1)

$\Rightarrow$ resolution is a K3 surface. (Rmk: can blow up T first & argue $\mathbb{Z}/2$ action lifts to blow-up, fixing exceptional divisors).

(for a general cplx. torus, this need not be projective).

& each nodal singularity gives a (-2) rational curve (called the nodal curves) (but they're smoothly embedded spheres; resolution of A_1 singularity)

Topology / Hodge theory

$\pi_1(X) = 0$.

Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1. & & \end{array}$$

can deduce via computing $\chi(X)$, $\chi_h(X)$, & Noether's formula.

The intersection form

$$(H^2(X,\mathbb{Z}), \cup) \cong (-E_8) \oplus (-E_8) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

rk. 22

even, i.e., $(x,x) \in 2\mathbb{Z}$ (b/c it's Spin; $c_1 = 0$, so w_2 vanishes).

Complex line bundles

Recall the exponential sequence of sheaves $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$.

$\rightsquigarrow$ induces in LES

$$H^1(X;\mathcal{O}_X) \longrightarrow H^1(X;\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \longrightarrow H^2(X;\mathcal{O})$$

(a) $H^1(X;\mathcal{O}_X) = 0$ for a K3

$H^1(X;\mathcal{O}_X^*) = \mathrm{Pic}(X)$ by definition; hence (a) $\Rightarrow \mathrm{Pic}(X) \hookrightarrow H^2(X;\mathbb{Z})$ for X a K3

(b) $H^2(X;\mathcal{O}) = H^{0,2}(X;\mathbb{C})$

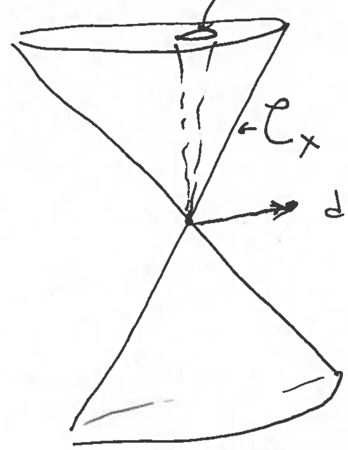
and, (b) $\Rightarrow \underset{\text{sub group}}{\mathrm{Pic}(X)} \hookrightarrow (H^{1,1}(X;\mathbb{R}), \cup)$ b/c fixed under conjugation.

Hodge index theorem: $\Rightarrow$ $\cup$ has signature $(1,19)$. ~~by~~

(uses fact X is kähler)

(some) Terminology: $d \in H^{1,1}$ ~~$\in H^2(X;\mathbb{Z})$~~ is divisonal (if it comes from a divisor); effective (comes from an effective divisor), irreducible (if it comes from an irreducible divisor); nodal (comes from a (-2) curve).

Picture: (of $H^{1,1}$)



Kähler cone $= \{x \mid (x,d) > 0 \ \forall\, d \text{ effective}\} \cap \mathcal{C}_X$; denote it $\mathcal{C}_X^+$.

Fact: If $x,y \in \mathcal{C}_X$, then $\langle x,y\rangle \geq 0$, (linear algebra/ Cauchy-Schwarz).

and if one is in the interior (of x,y), then $\langle x,y\rangle > 0$. (Think of $\mathcal{C}_X$ as closed; includes boundary).

Lemma 1: If $d \in \mathrm{Pic} \setminus \{0\}$, satisfies $(d,d) \geq -2$, then either d or $-d$ is effective.

Proof: $d \sim \mathcal{L}$ line bundle

The holomorphic Euler characteristic $\chi_h(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L})$

$$\leq h^0(\mathcal{L}) + h^0(\cancel{K_X} \otimes \mathcal{L}^\vee)$$

(by Serre duality) (b/c $K_X \cong \mathcal{O}_X$)

$$\underset{\text{(Riemann-Roch)}}{=} \frac{1}{2} d(c_1(X) + d) + \frac{\cancel{K_X^2}^{\,0} + \overset{\text{top Euler char.} = 24}{\chi(X)}}{12}$$

$$= \frac{1}{2}(d,d) + 2 \geq 1.$$

Lemma: If d is irreducible $\Rightarrow (d,d) \geq -2$.

Proof: By adjunction,

$$\underbrace{K_{\cancel{X}} + D\Big|_D}_{= D^2 \text{ (b/c } K_X = \mathcal{O}_X)} = K_D \geq -2.$$

(if D is smooth) ~~though lemma holds~~
(Lemma holds regardless).

Cor: The semigroup of effective classes is generated by nodal curves and integral points in $\mathcal{C}_X$.

(need to show that if have an irred. class, then nodal or in here ↗)

Pf: ~~If irred. class~~ d irreducible & $(d,d) > -2 \Rightarrow (d,d) \geq 0$ (b/c lattice is even).

Cor: $\mathcal{C}_X^+$ Kähler cone $= \{x \in \mathcal{C}_X \mid (x,d) > 0 \ \forall\, d \in \mathrm{Pic}$ w/ $d^2 = -2\}$.

2. Torelli theorems: rough Q: When are two K3s X, X' isomorphic?

If there exists a biholomorphism $f: X \xrightarrow{\sim} X'$, then induces

$$H^2(X,\mathbb{Z}) \xrightarrow[\cong]{\varphi} H^2(X';\mathbb{Z}).$$

1) φ preserves cup product
2) $\varphi_{\mathbb{C}}$ preserves Hodge decomp.

} say φ is a Hodge isometry

In fact, converse is true

3) effective $\rightsquigarrow$ effective } say φ is effective.

Thm: ("Torelli thm") [Shafarevich, Piatetski-Shapiro] If $\varphi: H^2(X;\mathbb{Z}) \to H^2(X';\mathbb{Z})$ is a Hodge isometry which is effective, it is induced by a unique biholomorphism.

If d is nodal, can define:

$$S_d: H^2(X;\mathbb{Z})^{\circlearrowleft} \quad x \longmapsto x + (x,d)\,d.$$

Because d is nodal & d is in $H^{1,1}$ (so orthogonal to $H^{2,0}$)
$d^2 = -2$

$\Rightarrow S_d$ is a Hodge isometry. (reflecting across plane orthogonal to d).

$S_d :=$ a Picard-Lefschetz reflection. Have an action

$$W_X = \{S_d \mid d \text{ is nodal}\} \circlearrowright \mathcal{C}_X.$$

Fact: $\mathcal{C}_X^+$ is a fundamental domain for this action.

$\phi: H(X,\mathbb{Z}) \longrightarrow H(X',\mathbb{Z})$, a Hodge isometry, is effective if and only if

Cor: (of earlier Cor). it maps one element of $\mathcal{C}_X^+$ into $\mathcal{C}_{X'}^+$.

Sketch: Just need to show ϕ(nodal class) is effective.
But either ϕ(nodal) or $-\phi$(nodal) is effective.
(The hypothesis that one elt. of $\mathcal{C}_X^+ \to \mathcal{C}_{X'}^+$ implies ϕ(nodal) is effective).
If $x \in \mathcal{C}_X^+$ has $\phi(x) \in \mathcal{C}_{X'}^+$ then $(\phi(x), \phi(\text{nodal}) = (x, \text{nodal}) > 0$
$\Rightarrow \phi$(nodal) is effective &
$-\phi$(nodal) is not.

All of this together implies

Thm (weak Torelli theorem): $X \underset{\text{bihol.}}{\cong} X' \iff \exists$ a Hodge isometry
$H^2(X,\mathbb{Z}) \xrightarrow[\varphi]{\cong} H^2(X',\mathbb{Z})$.

Pf: After applying some Picard-Lefschetz S_d's, can arrange any such φ becomes effective.

§3. Moduli spaces.

Call (X,ϕ), X a K3, $\phi: H^2(X;\mathbb{Z}) \xrightarrow[\text{isometry}]{\sim} L = (-2E_8) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (ϕ: marking).
a marked K3 surface.

The marking gives $\phi_{\mathbb{C}}: H^2(X;\mathbb{C}) \longrightarrow L_{\mathbb{C}}$.

Define

$$\tau(X,\phi) = [\phi_{\mathbb{C}}(\omega_X)] \in \mathbb{P}(L_{\mathbb{C}})$$

hol. volume form, well-defined up to scaling.

Actually $\tau(X,\phi) \in \Omega = \left\{ [\omega] \;\middle|\; \begin{array}{l} (\omega,\omega)=0 \\ (\omega,\bar{\omega})>0 \end{array} \right\} \subseteq \mathbb{P}(L_{\mathbb{C}})$

(note Ω has dim 20, b/c $L_{\mathbb{C}}$ has dim. 22, $\mathbb{P}(L_{\mathbb{C}})$ has dim. 21, & have imposed one equation).

This is called the $\boxed{\text{period map}}$. (the element $\tau(X,\phi)$ associated to (X,ϕ)) that is.

To set this up as an actual map:
~~Look at the~~

the Kuranishi space of X (the 'big deformation space') has tangent space.
(space of integrable J's mod diff.).

$$H^1(X;T_X) \cong H^1(X;\Omega_X^1) = H^{1,1}(X;\mathbb{C}) \quad \& \; h^{1,1}=20.$$

using $T_X \cong \Omega_X^1$ via the hol. volume form.

Now known in general:
X CY $\Rightarrow$ Kuranishi space is smooth (but simpler for X a K3), universal, dimension 20.

Call $\mathcal{M}_{(K3,\phi)}$ this moduli space of K3s w/ fixed markings. (point: K3s may have global automorphisms, so marking helps get rid of these; but the local Kuranishi picture works b/c maybe no local auts.), (ϕ is discretely more data).

The period map

$$\tau : \mathcal{M}_{(K3,\phi)} \longrightarrow \Omega \subset \mathbb{P}(L_{\mathbb{C}}) \quad \text{is a } \underline{\text{local isomorphism}}.$$

Given a marked family

$$\begin{array}{ccc} Y & \supset & Y_0 \\ \downarrow & & \downarrow \\ S & \ni & 0 \end{array} \quad \rightsquigarrow \text{period map } \tau : S \longrightarrow \Omega, ~~\text{period map}~~$$

and $T_\ell \Omega = \mathrm{Hom}(\ell, \ell^\perp/\ell)$.

Can compute $d\tau : T_0 S \longrightarrow T_{\tau(0)} \Omega$ as follows:

$$T_0 S \xrightarrow{KS} H^1(Y_0, T_{Y_0}) \xrightarrow{\sim} \mathrm{Hom}(H^{2,0}, H^{1,1})$$

Kodaira-Spencer map

|| (using marking ϕ)

$\mathrm{Hom}(\tau(0), \tau(0)^{\perp}/\tau(0))$ which $=$

(b/c have $H^1(T_{Y_0}) \otimes H^0(\Omega^2) \Rightarrow H^1(\Omega^1)$,

or dually, $H^1(T_{Y_0}) \xrightarrow{\sim} \mathrm{Hom}(H^0(\Omega^2), H^1(\Omega^1))$)

(using this, it's very straightforward to check

$\tau : \mathcal{M}_{(K3,\phi)} \to \Omega$ is a local iso.).

Facts: 1) τ is surjective (indeed, one can e.g. show Kummer surfaces are dense

(this is used to prove Torelli in fact; check it for Kummer & then a degeneration/"approximation" argument).

2) τ determines the Hodge structure.

$\Rightarrow$ all the fibers are isomorphic.

(weak Torelli)

In particular, $\tau^{-1}(p) \longleftrightarrow \left\{ \begin{array}{l} \text{chambers of action of } W_X \curvearrowright \mathcal{C}_X \\ \text{for any } X \text{ in the fiber} \end{array} \right\}$.

(or rather; $W_X, \mathcal{C}_X$ for any marked $(X;\phi)$ are in fact just determined by p!).

3) $\mathcal{M}_{(K3,\phi)}$ is not Hausdorff.

(in the limit of a family, a (-2) curve could pop out; & chamber could split).

E.g., "Atiyah flop" (two explicit families of K3's over disc which coincide away from origin.).