I. HMS for K3s (main ref: slides by Huybrechts)

N. Sheridan, 11/9/2016

K3 learning group

$X = K3$

$\widetilde{H}_{cpx}(X) := H^*(X;\mathbb{Z})$, $\langle \alpha,\beta\rangle := \langle -\alpha_0,\beta_4\rangle + \langle \alpha_2,\beta_2\rangle + - \langle \alpha_4,\beta_0\rangle$

wt-2 Hodge str. $\widetilde{H}^{1,1}_{cpx} = H^{1,1}\oplus H^0\oplus H^4$.

If $\phi : H^*(X;\mathbb{Z}) \xrightarrow{\sim} L\oplus U$ marking

$\widetilde{H}_{cpx}(X)$ det'd by cpx. period pt

$\phi_{\mathbb{C}}(H^{2,0}(X)) \in \widetilde{\Omega} := \{ k \in \mathbb{P}((L\oplus U)_{\mathbb{C}}) : (k,k)=0,\ (k,\bar{k})>0 \}$.

Torelli: X, Y proj. K3s

$$D^b(X)\cong D^b(Y) \iff \widetilde{H}_{cpx}(X)\cong \widetilde{H}_{cpx}(Y)$$

Note: to pin down X by Torelli $\iff$ find $H^2(X)\subset \widetilde{H}_{cpx}(X)$

$\iff$ find $H^0\oplus H^4 \subseteq \widetilde{H}^{1,1}_{cpx}(X) \subseteq \widetilde{H}_{cpx}(X)$

$\iff$ $U \subset (\text{period part})^{\perp}$.

Let $\omega_{\mathbb{C}} := i\omega + B$ be complexified Kähler form: ω Kähler, $B\in H^{1,1}(X;\mathbb{R})$

Define $\phi_{\mathbb{C}}(\exp(\omega_{\mathbb{C}})) \in \widetilde{\Omega}$

$$\| \\ 1+\omega_{\mathbb{C}}+\tfrac{1}{2}\omega_{\mathbb{C}}^2.$$

$\widetilde{H}_{Käh}(X,\omega_{\mathbb{C}}) :=$ corresp. wt. 2 Hodge structure.

(ind. of J, just $[\omega_{\mathbb{C}}]$-dependent).

Conj*: $\widetilde{H}_{cpx}(Y)\cong \widetilde{H}_{Käh}(X,\omega_{\mathbb{C}})$, Y proj., then $D^b(Y)\cong D^{\pi}Fuk(X,\omega_{\mathbb{C}})$?

...re, Fuk defined over $\mathbb{C}$ (Lot: convergence problems!) holom.
discs weighted by $\exp(2\pi i \int \omega_{\mathbb{C}}) \in \mathbb{C}$.
(Rmk: (a) $\int B$ is not an invt. of [u] unless $B|_L = 0$!!
(b) convergence?)

Q: $\widetilde{H}_{\text{käh}}(X, i\omega) \cong \widetilde{H}_{\text{käh}}(X', i\omega') \overset{?}{\Rightarrow} \text{Fuk}(X,\omega) \cong \text{Fuk}(X',\omega')$?

(Rmk: to have a n.c. der. Torelli, would need to know integral lattices match up.)

Note: $\widetilde{H}_{cpx}(Y) \cong \widetilde{H}_{\text{käh}}(X, \omega_{\mathbb{C}})$

$\Rightarrow H^*(Y;\mathbb{Z}) \cong H^*(X;\mathbb{Z})$

$\boxed{\text{nc der. Torelli}}$

$H^{2,0}(Y) \longleftrightarrow [\exp(\omega_{\mathbb{C}})]$

$H^0(Y) \oplus H^{1,1}(Y) \longleftrightarrow U \subset \exp(\omega_{\mathbb{C}})^{\perp}$ If no such U exists:

Y may be noncommutative ((K3, Brauer)).

In SYZ: $H^0(Y) \oplus H^4(Y) = \langle ch(\mathcal{O}_p), ch(\mathcal{L}_p t) \rangle$.

$\underset{\text{SYZ}}{\longleftrightarrow} \langle T^2, Tw_{\text{section}}(T^2) \rangle$.

Λ = Novikov field

Conj: X K3, ω = kähler form, $U \subset [\omega]^{\perp}$. Then, $\exists$ Λ-point

$P: \text{Spec}\,\Lambda \longrightarrow \Omega \subset \mathbb{P}(H^0 \oplus U^{\perp} \oplus H^4(X))_{\mathbb{C}}$ (take U, replace by $H^0 \oplus H^4$)

$\cong$

L.

need to "algebraize period domain." ~ messy.

"$T \longmapsto T^{\omega/2\pi i} := \exp(\omega_T)$"

"$1 + \omega_T + \omega_T^2$"

$\Rightarrow$ proj. K3 surface Y over Λ, and

$$D^\pi Fuk(X,\omega) \simeq D^b Coh(Y) \quad (\text{as } \Lambda\text{-linear categories.})$$

Conj: If have SYZ fibration, & finite collection of $\{L_i\}$,
then in some degenerate limit of 1-param. family of J near $\{L_i\}$, get convergence..

Answer to Q from before: [Smith-Sheridan]

$$(\tilde{H}_{Kah}(X,\omega) \cong \tilde{H}_{Kah}(X',\omega) \Rightarrow Fuk(X) \cong Fuk(X')?$$

Ans: Yes, but for a stupid reason:

$H^*(X) \cong H^*(X')$ integral cohomology

$$1 + \omega_T + \frac{\omega_T^2}{2} \longmapsto 1 + \omega_T' + \frac{(\omega_T')^2}{2} \quad \forall T.$$

"Λ period point!"

Now recall $\omega_T = (\log T)\frac{v}{2\pi i}$; by (*) & $1, \frac{\log T}{2\pi i}, \left(\frac{\log T}{2\pi i}\right)^2$ independent over $\mathbb{Q}$, it follows $\Rightarrow$ $H^0 \oplus H^4 \longrightarrow H^0 \oplus H^4$

$\Rightarrow \omega_T \longmapsto \omega_T'$, $1 \mapsto 1$, $\omega_T^2 \mapsto \omega_T'^2$ $(\log T)^?$

$\Rightarrow (X,\omega) \cong (X',\omega')$ using Torelli.
sympl.

(Space of period points identifying those are connected; // transport & identify)

(point: cannot have a 'line' in flat coordinates (complex) which goes to maximally unipotent monodromy)

Example: Kapustin-Orlov argue (in case of some abelian varieties (those w/ cpx. multiplication)) $K_0(D^bCoh(X))$ has "extra" classes (ch. lands in $\oplus H^{p,p}$, orthogonal to $H^{2,0}$.)
corresponding to "objects" of $Fuk(X^\vee,\omega^\vee)$ with homology class in $H^*(X^\vee)$ not of pure degree but orthogonal to $\exp(i\omega^\vee)$ = Kähler period point.

$\leadsto$ "coisotropic branes"? e.g. homology class in $H^0(X^\vee)$?

This doesn't happen in the corrected Fukaya category:

If $\alpha \perp (1 + \omega_T + \frac{1}{2}\omega_T^2)\ \forall T$,

then $\alpha \perp H^0$, $\alpha \perp H^4$ (by independence of terms again)

$\Rightarrow \alpha \in H^2$, $\alpha \in [\omega]^\perp$.

(might be an issue not over Λ_T; but in convergent ~~setting~~.
idempotent ~~summands~~ don't help, b/c b/c $\mathcal{OC} : HH_\bullet(\mathcal{F}) \to QH$ is graded ($\mathbb{Z}$-graded case) so always lands in H^2!!)

II: HMS for quartic (Seidel):

$$X = \{\textstyle\sum z^4 = 0\} \subset \mathbb{CP}^3, \quad Y_\lambda = \{z_1 \cdots z_4 = \lambda \textstyle\sum z_i^4\} \subset \mathbb{P}^3_\Lambda$$

$\lambda \in \Lambda$.

$$G := \ker\left((\mathbb{Z}/4)^4 / (\mathbb{Z}/4) \xrightarrow{\Sigma} \mathbb{Z}/4 \right).$$

roots of unity.

$\tilde{Y}_\lambda$

$\bar{Y}_\lambda := \tilde{Y}/G$. Has 6 A_3 singularities (when any two coordinates vanish).

$Y_\lambda :=$ resolution of $\bar{Y}_\lambda$. Smooth K3 $/\Lambda$.

Thm (Seidel): $\exists\ \lambda \in T + O(T) \in \mathbb{C}[[T]] \subset \Lambda$.

s.t. $D^\pi \mathrm{Fuk}(X, \omega_{FS}) \cong D^b\mathrm{Coh}(Y_\lambda)$.

(λ is unique in $\mathbb{C}[[T]]$, but can change by action of $\mathrm{Gal}(\Lambda / \mathbb{C}[[T]])$ (usual mirror map))

Maybe λ^4 is unique, b/c it's the ~~moduli~~ complex parameter on B-model moduli space