M. Abouzaid, Mini-course I.

Mirror symmetry: Curve counts weighted by area $\sum e^{-\langle [u],[\omega]\rangle}$ (*) ω varies in family.

Problem: These only converge a priori if have positivity $\Rightarrow$ finiteness of sum.

Instead, we use the Novikov field.

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; \begin{array}{l} a_i \in \mathbb{C} \\ \lambda_i \in \mathbb{R} \\ \lim_{i\to\infty} \lambda_i = +\infty \end{array} \right\}$$

Replace the series (*) by

$$\sum \_\, T^{\langle [u], \omega \rangle}$$

if ω varies in family, should get an analytic function...

References:

Rigid analytic: • Tate, Rigid analytic spaces

• Bosch, Lectures on formal and rigid geometry.

Symplectic: • Fukaya: Cyclic symmetry and Adic convergence

• Abouzaid: Family Floer cohomology and mirror symmetry

Key idea: replace the euclidean norm on $\mathbb{C}$ by the non-arch. norm on Λ

$$\left| \sum a_i T^{\lambda_i} \right| = e^{-\lambda_0}$$

λ_0 is the leading term

(Rmk: T^{10^6} is very small; T^{-10^6} very big)

In these lectures, everything will be expressed in terms of valuation.

$$\mathrm{val}\left(\sum a_i T^{\lambda_i}\right) = \lambda_0.$$

(note: $\mathrm{val}(0) = +\infty$).

Example 1: The unit disc.

$$D = \{ z \in \Lambda \mid \mathrm{val}\, z \geq 0 \}$$

(so norm ≤ 1).

(Rmk: D is both closed & open in Tate's "topology" ← in quotes b/c trouble w/ arbitrary unions.

Following Tate) $[\quad\quad)$

We define the ring of analytic functions on D

Difficult to define D directly as a ring of functions on a space, but see work of Berkovich.

to be

$$\Gamma_D = \left\{ \sum_{i=0}^{\infty} c_i z^i \;\middle|\; c_i \in \Lambda, \text{ which converges at all } z \in D \right\}$$

i.e., $\lim_{i \to \infty} \mathrm{val}(c_i z^i) = +\infty \;\; \forall z \in D$

$\mathrm{val}\, c_i + i\, \underset{\geq 0}{\underline{\mathrm{val}\, z}} = +\infty$

(Tate algebra in one variable)

[Rmk: convergence at $0 \overset{\text{forces}}{\Rightarrow}$ that only positive powers of z are allowed!]

$\iff \lim_{i \to +\infty} \mathrm{val}\, c_i = +\infty.$

In Floer theory, if end up with

$\sum c_i z^i$ ← "homology class of ∂u"

↑ contribution of hol. disc.

To check convergence: often can use Gromov compactness, which says $\exists$ only "finitely many" discs of bounded area. (rigid)

Appearance in symplectic topology –

X symplectic m'fold, J regular

$\cup$

orientable → L Lag'n, and L bounds no J-holo. discs.

We know that the "simple" objects of $\mathcal{F}(X)$ which are supported on $L \rightsquigarrow b$ "bounding co-chains",

i.e., an odd-dimensional $H^*(L)$ element (if L bounds no discs).

Assume $H^1(L) = 0$. Then, bounding co-chains are elements of $H^{odd}(L; \Lambda)$ which lie in the unit disc $H^{odd}(L; \Lambda_0)$ (val ≥ 0)

↑ (basically D, but thought of algebraically)

e.g., $L = S^3 \subset T^*S^3$.

Then, $H^{odd}(S^3, \Lambda) = \Lambda \supset D$ ← unit disc.

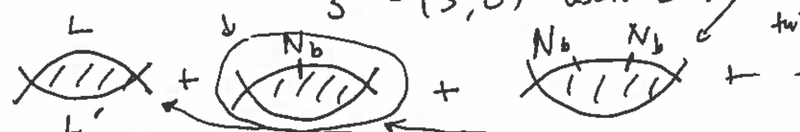
$S^3 = (S^3, 0)$ usual S^3.

Q: why not $(S^3, T^{-5}[S^3])$?

$b \sim N_b \subset L$ cycle.

If L' another Lagrangian, compute

$HF^*(L', (L, b))$:



note since $H^1(L) = 0$, $\mathrm{codim}(N_b) > 1$ & these are not same (cut down!).

Since $\mathrm{val}(b) \geq 0$, ~~this~~ this expression still converges. (that's the point)
$(b \in D)$

Rmk: • if $\lambda = 0$, so $\mathrm{val}(b) = 0$, still ok (borderline case!).

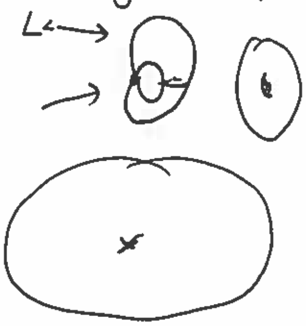
(putting all b constants in cuts down the entire Gromov compact moduli space of all discs below fixed energy at once, so ~~by all b's~~ preserves compactness)

so  converges w/ $\mathrm{val} = 0$.)

needs $H^1(L) = 0$ so all counts different

• if $b \in H^{\mathrm{even}}(L)$, lose $\mathbb{Z}/2$ grading (which get when L is orientable)

Other example: Lag'n torus fibration with nodal singularity.



$HF^\bullet(L,L)$
$\cup$
1-dim'l vector space spanned by a generator corresponding to one of the two self-intersection points, call this generator b

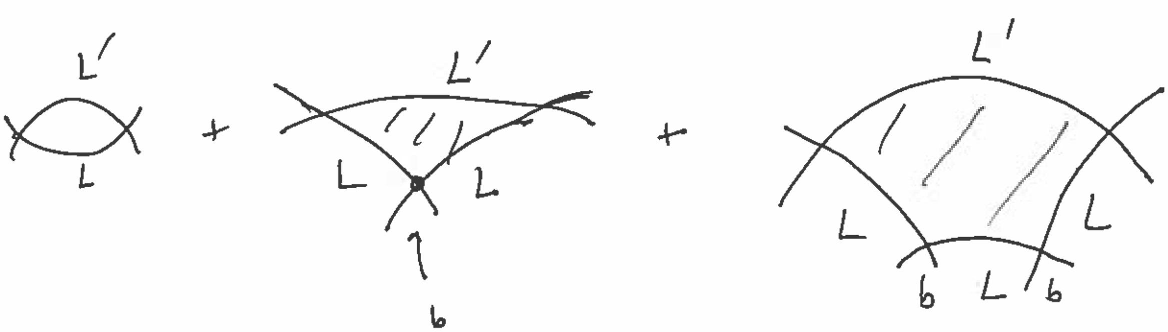
"(L,b)" makes sense again assuming $\mathrm{val}\, b \geq 0$

(by work of Fukaya, Cieliebak, Ekholm, ...)

c.f. [Fukaya 2009] lecture notes at Berkeley

[Cieliebak-Ekholm-Latschev] paper; assume local form, J standard.

The subtlety: in $HF^*(L', (L,b))$



proving cpctness for these doesn't follow from same exact techniques...

Example 2: Pick $a, b \in \mathbb{R}$

$[a,b] \subset \mathbb{R}$

$A(a,b) = \{ z \mid a \leq \mathrm{val}\, z \leq b \}$.

Functions on $A(a,b)$ are convergent Laurent series :

$$\left\{ \sum_{i=-\infty}^{+\infty} c_i z^i \;\middle|\; \forall\, z \in A(a,b), \lim_{i \to \pm\infty} \operatorname{val} c_i z^i = +\infty \right\}$$

Given λ with $a < \lambda < b$, the growth of $\operatorname{val} c_i z^i$ at z such that $\operatorname{val} z = \lambda$ is faster for $i < 0$ than

$$c_i + i a.$$

Need:

$$\lim_{i \to +\infty} \operatorname{val} c_i + i a = +\infty$$

$$\lim_{i \to -\infty} \operatorname{val} c_i + i b = +\infty \;..$$

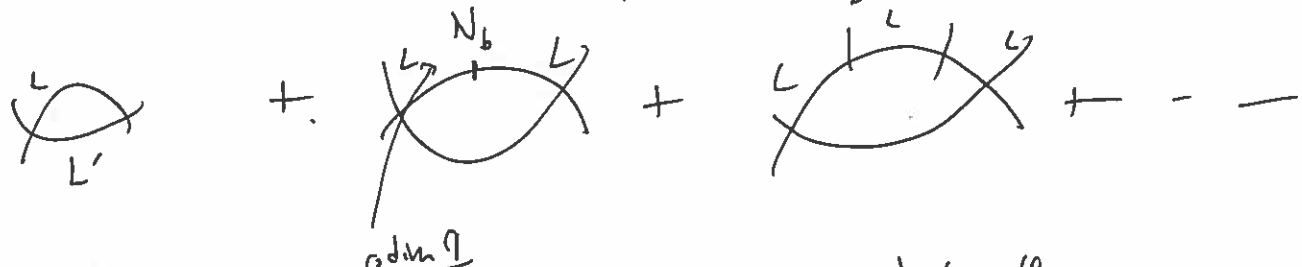
Let's define: $x_1 = T^a z$, $x_2 = T^{\cancel{-b}\;-a} z^{-1}$

We can think of the ring of functions as series in x_1 & x_2 mod relations $x_1 x_2 = T^{\cancel{a-b}\; b-a\,?}$.

In sympl. topology, the case $\underline{\underline{a=b=0}}$ (i.e., $\lim_{i \to \pm\infty} \operatorname{val} c_i = +\infty$)

corresponds to "bounding co-chains" in $H^1(L)$, $b \in H^1(L;\Lambda)$.

Try to implement bounding co-chains as before: namely



[FOOO].

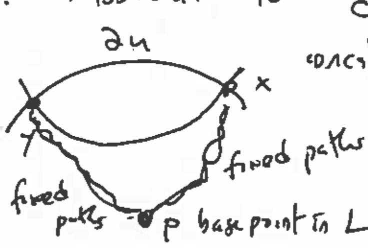
These conditions do not cut down space, b/c N_b don't cut down the space..

but, be careful to make it work.

[Cho, FOOO]: Associate to ∂u a class in $H_1(L, \mathbb{Z})$)

concat. w/ ∂u. fixed paths.

Define the "twisted" diff'l associated to $\tilde{b} \in H^1(L, \Lambda^*)$ non-zero



$$\sum_u \langle \tilde{b}, [\partial u] \rangle T^{E(u)}$$

negative powers in general.

this $\tilde{b}$ is basically formally e^b from before; all those terms become

If we assume that

$$“\mathrm{val}(\tilde{b}) = 0”, \text{ i.e., } \quad b \in H^1(L, U_\Lambda)$$

↑ subgroup $\subset \Lambda^*$ w/ val $c = 0$.

get: $\sum (a(u) + \text{higher order} \ldots) T^{E(u)}$, so

convergence assured.

⇒ the set of bounding co-chains in $H^1(L)$ is a product of annuli of width 0.

(Rmk: can't have “$\mathrm{val}(\tilde{b}) > 0$”, doesn't make sense, b/c need to test against classes $[\gamma], -[\gamma] \in H_1(L)$ in principle,) .

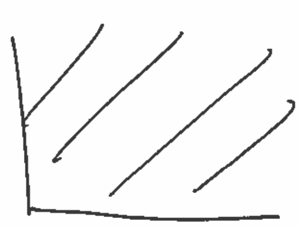
so produce arbitrary valuations in sum,

General theory:

Tate algebra → $T_n = \Lambda\langle\langle x_1, \ldots, x_n \rangle\rangle$

Formal series in n variables which converge on the product of n copies of D; (i.e., if we evaluate at $\{x_i\}$ s.t. $\mathrm{val}\, x_i \geq 0 \ \forall i$)

condition on valuations:



upper quadrant..

Thm (Tate): T_n is Noetherian.

An affinoid algebra is the quotient of T_n by an ideal (which by above is necessarily finitely generated, of the form: $\hat{I} = (f_1, \ldots, f_k)$) .

The ~~[illegible]~~ affinoid domain associated to

$$T_n \twoheadrightarrow A$$ is the zero locus in $(D)^n$ of the corresponding ideal.

Ex: $T_2 = \Lambda\langle\langle x_1, x_2\rangle\rangle$ with ideal $x_1 x_2 = T^{\lambda}$.

in valuations: $\lambda_1 + \lambda_2 = \lambda$

in D^2: