M. Abouzaid, IAS Mini-Course Lecture 2 9/23/2016

Notation: Λ Novikov field

$\Lambda \supset \Lambda_0$ elements of val ≥ 0 (norm ≤ 1) subring

$\Lambda^* \subset \Lambda$ invertible elements (val $\neq 0$)

$\Lambda^* \supset U_\Lambda \subseteq \Lambda_0$

U_Λ: unitary elements (val $= 0$)

$\Lambda \sim \mathbb{A}^1$

$\Lambda_0 \sim D$

$U_\Lambda \sim A(0,0)$

$\Lambda^* \sim A(-\infty, +\infty)$

Given $L \looparrowright M$ immersed oriented Lag's with ⋔ double points, and bounds no hol. discs for some J or 1-gons ("tear-drops")

$J \rightsquigarrow HF^*(L,L)$ can be expressed in terms of $L \times_M L$

$L \times_M L \supset L \cup \{\text{double points}\}$

OR

$$HF^*(L,L) \cong H^*(L) \oplus H^*(\text{double points})\left[\begin{matrix}\text{this}\\ \text{sign}\\ \delta\end{matrix}\right]$$

$\mathbb{Z}_2$ graded

assign 0 to pos. intersection, $\delta = 1$ to neg. intersection

(corrected by sign coming from dimension)

(assuming no holomorphic bigons either!)

Last time: The "moduli space" of simple objects of $\mathcal{F}(M)$ with support on L is:

$\mathbb{Z}_2$ graded

$$H^1(L, U_\Lambda) \times H^{\text{odd}>1}(L, \Lambda_0) \times H^{\text{odd}}(\text{double points}, \Lambda_0[\delta])$$

↑ means, if degree of double point is odd, get Λ_0.

geometrically:

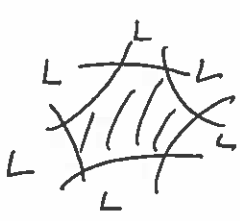
$$Y(L) = \underbrace{A(0,0)^{b_1} \times D^{b_{odd}-b_1}}_{\text{Gromov compactness}} \times \underbrace{D^{\#\text{odd double points}}}$$

why?

- Fukaya 2009 Berkeley lecture
- Cieliebak - Ekholm - Latschev

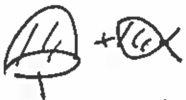
saying: "it takes some energy to cross any double point."

Note: need to exclude all holo. polygons

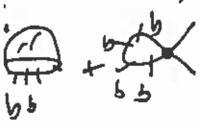


to get all of $H^{\text{odd}}(\text{double pts.}, \Lambda_0[\delta])$ in moduli space.

Slogan: $Y(L)$ is a rigid analytic space. If we drop conditions on no holo. discs, then we should get:



For $b \in Y(L)$,



① Equations on this space imposing the condition that a certain curvature $m_0 \equiv 0 \pmod{H^0(L)}$ "weakly unobstructed"

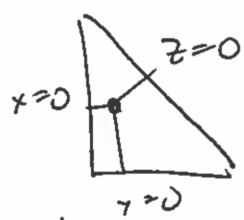
"zeros of section of $Y(L)$ to even cohomology"

② Equations imposing the condition that the homology of induced differential is non-zero,

(equivalent to $e_L \in H^0(L)$ not being in image).

Ex: (Cho, Cho-Oh, FOOO) Toric varieties

e.g., $\mathbb{CP}^2$ with area 1.



$L(a,b)$ $0 \le a$, $0 \le b$ area of disc w/ ∂ on $L(a,b)$ intersecting lines once.

Get an equation for "m_0" of this torus (function on $H^1(L, U_\Delta)$ of $(x,y) \in$)

$$W = T^a x + T^b y + \frac{T^{1-a-b}}{xy} \in H^0(L(a,b)) \cdot \; \; U_\Lambda^2 .$$

Since lives here, all objects ~~we prefer~~ satisfy ①.

② Get non-zero HF^* iff (x,y) is a critical point in variables (x,y) with (a,b) fixed,

$$\partial_x W = T^a - \frac{T^{1-a-b}}{x^2 y}$$

$$\partial_y W = T^b - \frac{T^{1-a-b}}{x y^2} .$$

Need a solution w/ $\mathrm{val}(x) = 0$, $\mathrm{val}(y)$ "arguing w/ valuations, as"

$\Longrightarrow$ $a = 1 - a - b$, $b = 1 - a - b$ (b/c otherwise cannot cancel)!

$\Rightarrow a = b = 1/3$.

(otherwise, find 3 points which live outside set).

<u>Key observation</u>: We can identify the potential W for $L(a,b)$

$(W: A^2 \to \Lambda)$ as the restriction of W for $L(\frac{1}{3},\frac{1}{3})$ to the product of annuli in $(\Lambda^*)^2$

$\uparrow$ bi-annulus

with bi-radii given by $\operatorname{val} x = a - 1/3$, $\operatorname{val} y = b - 1/3$. (both have thickness 0).

W is defined on A^2,

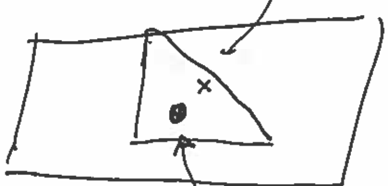
<u>but it extends naturally to</u> $(\Lambda^*)^2$. Q: what does the extension mean?

Can recover the extension for the other torus fibers from this single function.

$$(\Lambda^*)^2 \xrightarrow[\text{computed on } (1/3,\,1/3)]{W} \Lambda$$

$\downarrow \operatorname{val}$

let's say crit. point lies here at $L(a,b)$.



$L(1/3, 1/3)$

Then $L(a,b)$ would be equipped w/ a non-unitary local sys. w/ non-vanishing HF^*.

<u>But</u>: also need to impose

$0 \le a$

$0 \le a$

$a+b \le 1$

So, critical points away from this triangle are not <u>physical</u>.

(For Fano toric varieties, all critical points lie in  !) ..

& this procedure always gives a Laurent polynomial.

- FOOO: For any toric variety: W is an <u>analytic function</u> on domain inside $(\Lambda^*)^n$

given by $\operatorname{val} z \in \mathring{P}$ ← interior of moment polytope.

(little tricky, b/c basic building blocks are <u>closed</u> polytopes, so need to exhaust this by ~~the~~ closed polytopes)

Rmk: this only includes $H^2(L)$ part of moduli space

there's $\overset{\text{nb. in}}{H^{>2}}(L)$ not seen by this W function.
Λ

? (non-Fano case: W not a polynomial; ~~what's~~ not sure what the radius of convergence is?)

In this example, all isolated critical points.

(there are more interesting examples though)

Floer-theoretic origin of affinoid rings:

Assume L is embedded (no discs).

$Y(L) \sim H^1(L, U_\Lambda) \times H^{odd \geq 1}(L, \Lambda_0)$

The based loop space of L is ~~a~~ ~~[illegible]~~ group, so

$H_*(\Omega L; \Lambda)$ is a graded algebra.

Consider: $\hat{H}_*(\Omega L; \Lambda)$ completion with respect to the T-adic topology,

i.e., elts. $\sum_i T^{\lambda_i}\alpha_i$ $\boxed{\lim \lambda_i \to +\infty}$

(no length completion here)

(Rmk: $\widehat{H_*(\Omega L; \Lambda)} = H_*(\widehat{C_*(\Omega L; \Lambda)})$ in this case; ~~[illegible]~~

$\mathbb{Z}$-graded case: space of $\overset{(\mathbb{Z}\text{-graded})}{\text{rank } 1}$ modules over $H_*(\Omega L)$ $\longleftrightarrow$ $H^1(L, \Lambda^*)$

$\cap$ $\qquad\qquad\qquad \cup$

space of modules over the completion $\hat{H}_*(\Omega L; \Lambda)$ $\longleftrightarrow$ $H^1(L; U_\Lambda)$

ex: $H_0(\Omega L, \Lambda) \to$ Rep of $\pi_1(L)$
$\qquad\qquad = H^1(L; \Lambda^\times)$ monodromy.

Ex: $L \cong T^n$. Then,

$$H_*(\Omega L;\Lambda) \simeq H_0(\Omega L,\Lambda)$$

$$\cong \Lambda[z_1^{\pm}, \dots, z_n^{\pm 1}]$$

$$\cong \Lambda[H_1(L,\mathbb{Z})].$$

Functions on $A(0,0)$

$\sim$ Laurent series $\sum c_i z^i$

s.t. $\lim_{i\to\pm\infty} \mathrm{val}\, c_i = +\infty$.

$\cong$

completion of Laurent polynomials

Thus, $\hat{H}_*(\Omega T^n, \Lambda) \cong \Lambda\langle\langle z_1^{\pm 1}, \dots, z_n^{\pm 1}\rangle\rangle$

(ring of fxns. on $A_{z_1}(0,0)\times \dots \times A_{z_n}(0,0)$

$\cong$

$H^1(L, U_\Lambda)$).

Similar story in the $\mathbb{Z}_2$ graded case where

$H^1(L,\Lambda^*) \times H^{odd>1}(L,\Lambda)$ is a space of rank 1 odd modules over $H_*(\Omega L)$.

complete this $\Rightarrow$ restrict to $H^1(L,U_\Lambda)\times H^{odd>1}(L,\Lambda)$.

How does this work?

A rank 1 module is an augmentation

ev

$$H_k(\Omega L) \xrightarrow{\times[S^1]} H_{k+1}(\Omega L \times S^1) \xrightarrow{ev} H_{k+1}(L) \xrightarrow[b\in H^*(L)]{\langle\,,b\rangle} \Lambda$$

(Rmk: probably should have put "A_∞ modules, etc."; need higher multiplicative terms here), coming probably from "Chen's iterated integrals").

For S^3,

$$H_*(\Omega S^3) \cong \Lambda[u], \deg u = 2$$

Want to see that picking $b \in H^3(S^3, \Lambda)$ corresponds to the module

$$\Lambda_b \cong \Lambda[u] / (u - b = 0)$$

$$u : S^2 \longrightarrow \Omega S^3$$

$$\Rightarrow S^2 \times S^1 \longrightarrow S^3 \quad \boxed{\text{degree } 1}$$

Given $b \in H^3(S^3)$, action of u = multiplication by b.

$$\Rightarrow (S^3, \underset{\underset{H^3(S^3)}{\cap}}{b[S^3]}) \text{ gives the module } \Lambda_b.$$

Cor: The structure sheaf on $Y(L)$ comes from

$$\hat{H}(\Omega L)$$

(little delicate, b/c this is in general non-commutative).
For products of odd spheres (when $H_*\Omega L$ commutative), this is true on the nose)
(works b/c S^1, S^3, are groups & S^{2k+1} are rationally groups).

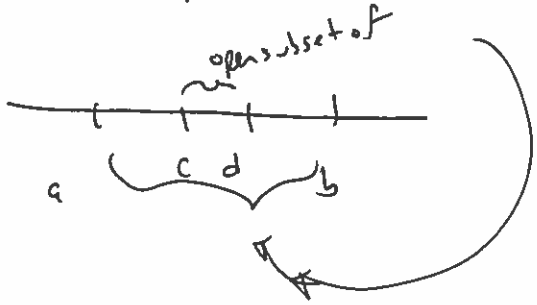
- Before, we interpreted m_0 as an assignment of a "curvature" to each $b \in Y(L)$.
- In terms of $\hat{H}(\Omega L)$, the curvature becomes an element of this ring.

This is given by

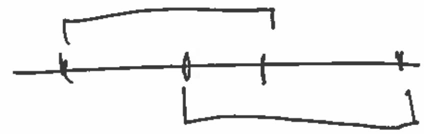
$$\boxed{\left[\sum [ev_0]_* \left[\mathcal{M}(\beta)\right]\right] \cdot T^{E(\beta)}}$$ only makes sense after completion.

Domains $\longleftrightarrow$ functions & cut down domains (Zariski topology).
Instead, can use the finer Tate's "G-topology".

In Tate's topology:



~~works~~ helps in symplectic topology to glue together



strictly not a Grothendieck topology, but good enough.

Rmk: ~~maybe one can~~ one could think of this as a stack of chain / group action,

by passing to H^*, we've picked a slice.