J. Pardon, Mini-course Lecture 1: 10/5/2016

<u>Liouville sectors and Fukaya categories of Stein manifolds</u>

(joint w/ Sheel Ganatra and Vivek Shende).

<u>Goal</u>: Localize Floer theory on (certain) Liouville manifolds.

<u>Conjecture</u> (Kontsevich): X Stein manifold, $\phi: X \to \mathbb{R}$ exhausting J-convex Morse.

$$\rightsquigarrow \mathbb{L} := \bigcup_{p \in \mathrm{crit}(\phi)} \text{descending m'flds}(p).$$

"= singular Lag'n spine." (though may actually be isotropic)

<u>Ex</u>: $X = T^*Q$, $\mathbb{L} = Q$.

$X = T^2 - pt.$, $\mathbb{L} = S^1 \vee S^1$.

Then, "$\mathrm{Fuk}(X) \simeq \mu\mathrm{Sh}\,\mathbb{L}$."

"sheaves on the smooth part, w/ something involved happening at singular points."

<u>Def</u>: Let X be a Liouville m'fld w/ boundary,

(i.e., (X,λ) is exact symplectic mfld w/ boundary, near ∞ modeled on the symplectization of a contact mfld w/ boundary).

Say X is a <u>Liouville sector</u> iff it satisfies the following equivalent conditions:

① $\exists$ a Ham. vector field X_I (<u>linear at ∞</u>) which is ~~∂X~~ $\pitchfork$ to ∂X.

in terms of I:

② $\exists$ $I: \partial X \to \mathbb{R}$ linear at ∞ ($ZI = I$ near ∞), &

$$dI|_{\text{char fol}(\partial X)} > 0.$$

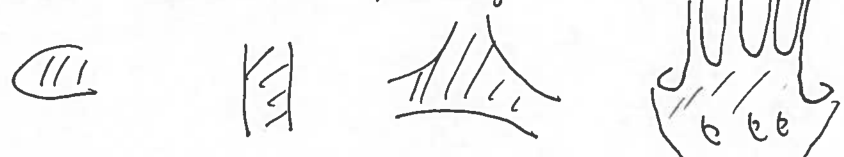
(so X_H commutes w/ Z)

($\Rightarrow$) the contact m'fld $\partial^\infty X$ has <u>convex</u> boundary). (equivalent to $X_H \pitchfork \partial X$)

<u>Examples</u>: • Any T^*Q (Q compact, <u>possibly with boundary</u>). • Any Liouville manifold.

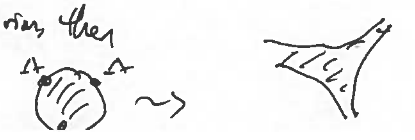
(take a vector field on Q $\pitchfork$ to ∂Q & lift to T^*Q, where it's Hamiltonian.

• Lots of Riemann surfaces w/ boundary:



; they are just not allowed to have a <u>circle</u> on the boundary...

• If X is a Liouville sector, $\Lambda \subseteq \partial_\infty X$ closed Legendrian, then "$X \setminus N_\varepsilon \Lambda$" is a Liouville sector. ex:



• can think of T^*Q as (arising in the above way) $(T^*\mathring{Q}$, canon. Liouville str., $\Lambda = \partial Q)$.

$\uparrow$ zero section in $T^*\partial Q$.

• (Liouville sector) $\times$ (Liouville sector) = Liouville sector (modulo rounding corners), which doesn't really matter here.

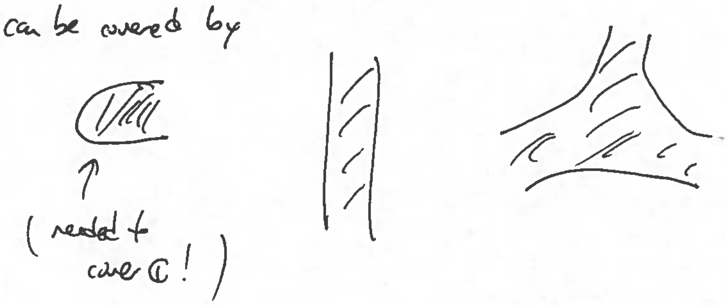
Rmks:- the boundary of a Liouville sector cannot be compact.

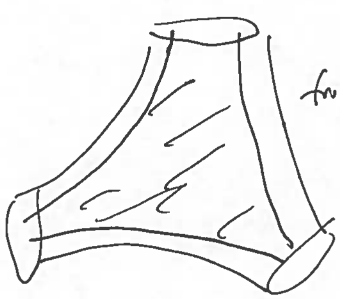
Liouville manifolds often admit interesting covers by Liouville sectors.

Ex: T^*Q (Q closed) can be covered by many $T^*(\mathrm{Ball})$ s.

(Rmk: in fact Ball $\cap$ Ball has corners, but the precise set-theoretic nature of the intersections is not relevant to our argument).

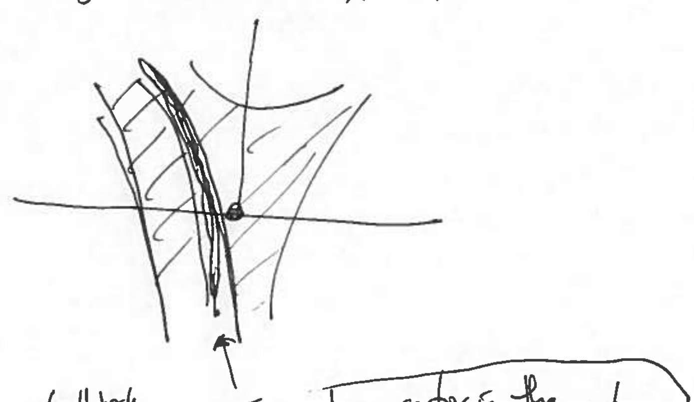
Any Riemann surface can be covered by





front, back overlaps.

In reality: our covers need to cover overlaps:



We only consider inclusions of Liouville sectors

$X \hookrightarrow X'$ which are proper & cylindrical (pull back Liouville form on the nose) near ∞.

at ∞.

(contrast this with Viterbo functoriality:

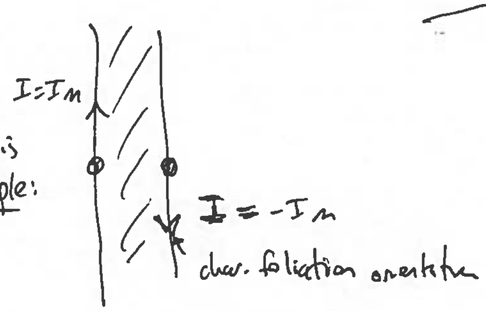
$D_1 \hookrightarrow D_2$ inclusion of Liouville domains, then

$\widehat{D_1} \hookrightarrow \widehat{D_2}$ (almost) never proper, yet have Viterbo restrictions [Viterbo, Abouzaid-Seidel]).

Heuristic reason for why we need to impose ① & ②:

Can SFT neckstretch along ∂X $\rightarrow$ get curves asymptotic to closed characteristics of boundary: but condition ② $\Rightarrow$ there are no closed characteristics on boundary):

ex: what is I in this example:



(rmk: might need to extend I to $O_p(\partial X)$ to get X_I, but ⋔ to boundary condition only involves $I|_{\partial X}$).

Conditions:

① & ② are

$\Longleftrightarrow$ ③ $\exists$ identification (near ∂X) $(X, \lambda) \simeq (\mathbb{C}_{Re \geq 0} \times W, \lambda_{\mathbb{C}} + \lambda_W + df)$, $f: \mathbb{C} \times W \to \mathbb{R}$ (f cptly supported)

w/ $I \longleftrightarrow Im_{\mathbb{C}}$. where W is a Liouville manifold

N.B. ③ $\Rightarrow$ the notion of a Liouville sector is very closely related to Z-Sylvan's notion of a stop on a Liouville manifold.

③ $\Rightarrow$ (get $\pi: O_p \partial X \longrightarrow \mathbb{C}$)

If choose J s.t. π is (J, j) holomorphic, can now constrain hol. curves.

(Rmk: flow of char. foliation is a proper $\mathbb{R}$-action, b/c $dI > \varepsilon$ bounded above & I linear near ∞; also note $\Rightarrow$ I takes all values.)

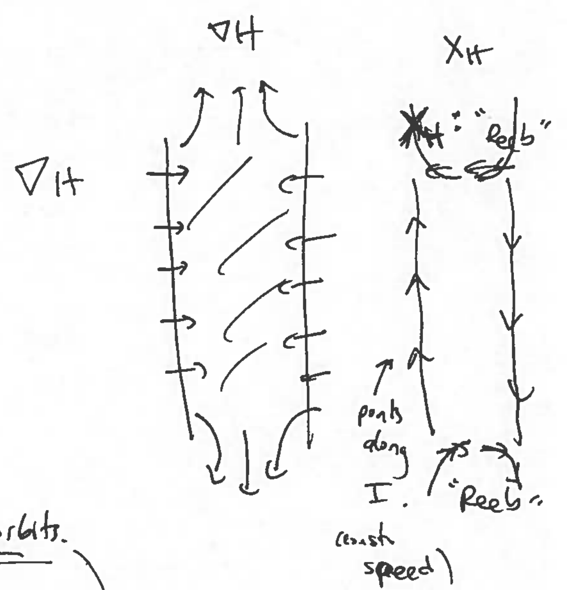
Floer theory on Liouville sectors:

- $\mathcal{W}(X)$ wrapped Fukaya category

- $SH^\bullet(X, \partial X) = \varinjlim HF^\bullet(X; H)$

$\exists$ natural map i: $H^\bullet(X, \partial X) \to SH^\bullet(X, \partial X)$

$H: X \to \mathbb{R}$ s.t. $H|_{O_p \partial X} = Re\,\pi$.

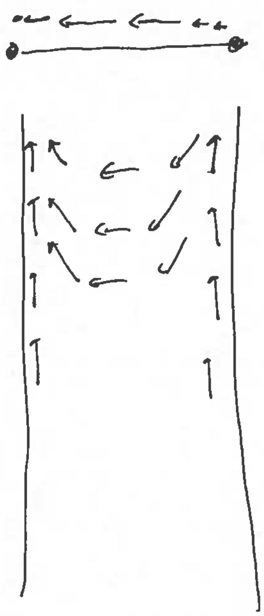


& i is an isomorphism if $\exists$ cutoff Reeb vector field w/ no closed orbits. (e.g., $T^*(Ball)$).

Def: On a contact manifold Y with boundary, a cutoff Reeb flow is the contact vector field for a contact Hamiltonian $H: Y \longrightarrow \mathbb{R}$, which vanishes to order 2 along ∂Y (otherwise >0).)

(Rmk: this is the Lie algebra of (contactomorphisms fixing ∂Y)).

Ex:

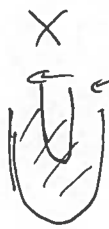


These are both <u>covariantly</u> functorial for $X \hookrightarrow X'$

& $\mathcal{OC}: HH_{n-*}(\mathcal{W}(X)) \longrightarrow SH^*(X, \partial X)$

$\mathcal{W}(X)$

<u>Examples:</u>



← only allowed L (needs to be properly embedded, <u>can't</u> touch ∂X!

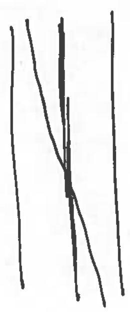
& L can be displaced from itself by flow + cptly supported flow

$\Rightarrow L \sim 0$.

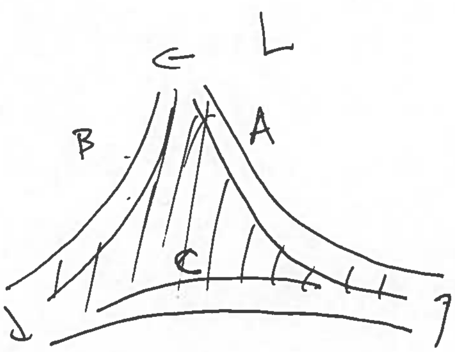
$\mathcal{W}(X) \simeq 0$ $\qquad$ $SH^*(X, \partial X) = 0$ here.

SH

0



$\mathrm{Rep}(\bullet) \simeq \mathcal{W}(X)$ $\qquad$ $\mathbb{Z}$



$\mathrm{Hom}(A,B) = \mathbb{Z}$ $\qquad$ & $\mathrm{Hom}(B,A) = 0$ etc --

$\mathrm{Hom}(B,C) = \mathbb{Z}$ $\qquad$ "univ. exact triangle"

$\mathrm{Hom}(C,A) = \mathbb{Z}$

$\mathrm{Rep}(\bullet \longrightarrow \bullet)$. $\qquad$ $\mathbb{Z}^{\oplus 2}$.

In these cases, look at the PSS maps:

$$\begin{array}{ccc} H^*(L) & \xrightarrow{PSS} & HW^*(L,L) \\ \downarrow i_! & & \downarrow \mathcal{OC} \\ H^{*+n}(X,\partial X) & \xrightarrow{PSS} & SH^{*+n}(X,\partial X) \end{array}$$

In particular, using theirs, the local $\mathcal{OC}$ maps are isomorphisms.

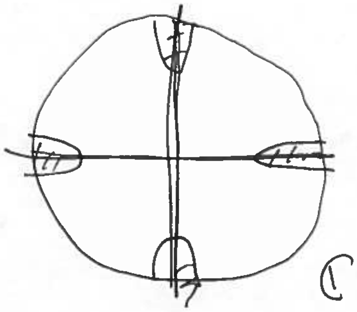
Another example:

$\mathbb{R}^n \cup i\mathbb{R}^n \subseteq \mathbb{C}^n$. gives a Liouville sector

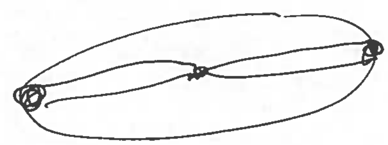
by looking at $\mathbb{C}^n \setminus N_\varepsilon(\partial_\infty(\mathbb{R}^n \cup i\mathbb{R}^n))$

(Zack: can look at quadratic Lefschetz fibration w/ two stops

$$\begin{array}{c} \mathbb{C}^n \\ \downarrow \Sigma z_j^2 \\ \mathbb{C} \end{array}$$



$\mathbb{C}^n$



two stops

Thm: Let X be a Liouville manifold, let $\{X_\sigma\}$ be a collection of Liouville sectors $\subseteq X$ indexed by a poset Σ. (e.g. T^*Q covered by T^*Balls).

Let $\mathcal{F}_\sigma \subseteq \mathcal{W}(X_\sigma)$ be full subcategories w/ $\mathcal{F}_\sigma \subseteq \mathcal{F}_{\sigma'}$.

If the local $\mathcal{OC}: HH_{*-n}(\mathcal{F}_\sigma) \to SH^*(X_\sigma, \partial X_\sigma)$ (just can represent 1 as hypercover Čech!) are isos, & $\{X_\sigma\}_{\sigma\in\Sigma}$ is a homology hypercover of X,

then $\mathcal{OC}: HH_{*-n}\left(\bigcup_{\sigma\in\Sigma} \mathcal{F}_\sigma\right) \to SH^*(X)$ hits 1.

Then [Abouzaid] $\Rightarrow$ $\bigcup \mathcal{F}_\sigma$ split-generate $\mathcal{W}(X)$.

"Homology hypercover" means: $\left\{\bigoplus_{\sigma_0 \leq \dots \leq \sigma_p} C^{BM}_*(X_\sigma)\right\} \to C^{BM}_*(X)$ should hit 1.