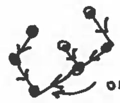
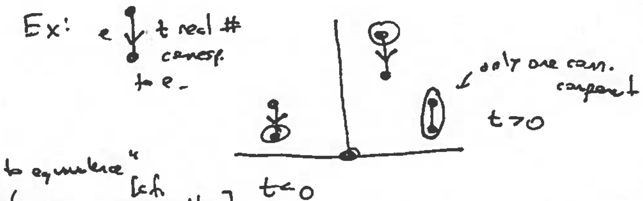
J. Pardon, Mini-course lecture 2, <u>Liouville sectors and Fukaya categories of Stein manifolds</u> 10/7/2016

Arboreal singularities (Nadler):

T - rooted tree 

$$\rightsquigarrow L_T := \left\{ \begin{array}{l} \{t_e \in \mathbb{R}\}\ e \in E(T). \\ x \in \pi_0(T \setminus \{e \mid t_e < 0\}) \end{array} \right\}$$

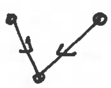
↑ conn. component, remove edges where t's are negatively labeled.
"choice of vertex, up to equivalence" [cf. Nadler].



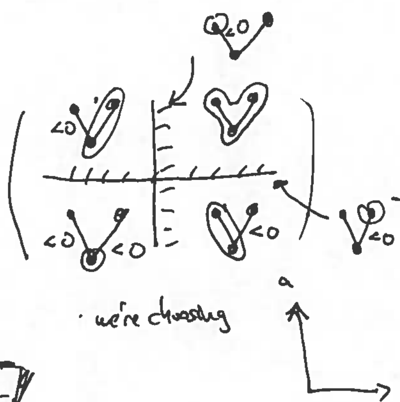
$$\downarrow$$

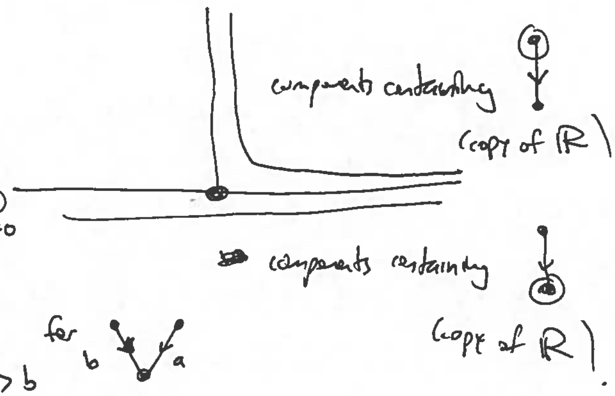
$$\mathbb{C}^{E(T)}$$

$$\text{sending } (\{t_e\}, x) = \left\{ \begin{array}{ll} i t_e & \text{if } t_e<0 \text{ and } e \text{ is on the path from } x \text{ to root of } T \\ t_e & \text{else} \end{array} \right\}$$

Ex: T = 

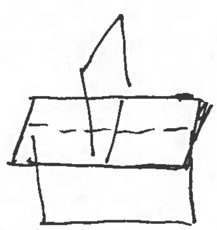
$\rightsquigarrow L_T = \mathbb{R}^2 \cup$ "1/2 conormal":





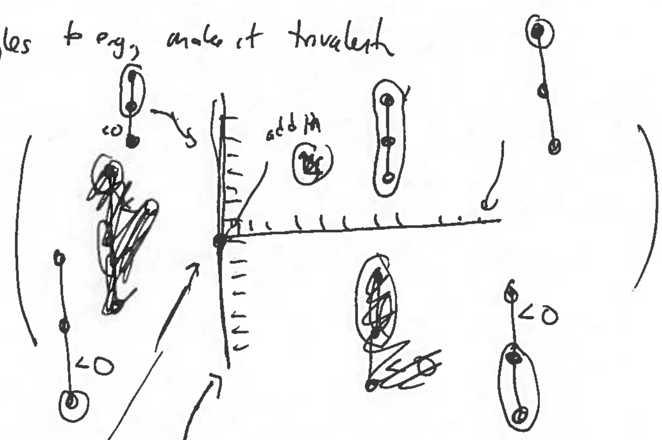
$\cap$

$\mathbb{C}^2 = T^*\mathbb{R}^2$

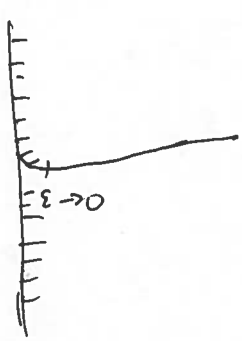


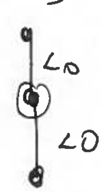
Rmk: there are three angles, +1, -1, -i; can choose other angles to e.g. make it trivalent.

Ex: T =  $\rightsquigarrow L_T = \mathbb{R}^2 \cup$ conormal



strictly speaking need to smooth: at this point



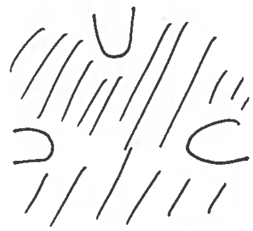


Let X_T be the Liouville sector $\mathbb{C}^{E(T)} \setminus N_\delta(\partial_\infty L_T)$

(Rmk: to be closed under products, strictly speaking should work w/ disconnected trees)

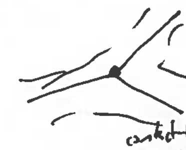
ex: 

L_T



X_T

Claim: the construction of producing a sector in complement of Λ works when Λ is singular/"subanalytic", if transverse to Reeb).



to produce N_δ, need to know it contact form $\boxed{\lambda - dt}$ has "convex" nbhd w/ boundary" (in this case, work an explicit one)

t: Reeb direction

Abuse notation:

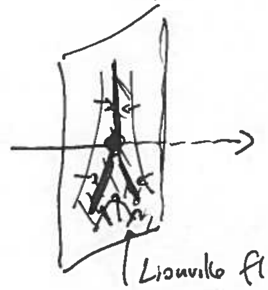
$$X_T^{2n} = X_T \times (T^*[0,1])^{n-E(T)}.$$

$\uparrow$

($\mathbb{C}$, w $\mathbb{R}$ as spine)

Another picture:

$t=0$ (Liouville).



Reeb.

Liouville flow

Rmk: $\partial_\infty L_{\bullet - \bullet}$ ⦶, $\partial_\infty L_{\bullet}$ = ⦶ .

Prop: [Ganatra - P. - Shende]:

$\mathcal{W}_T \hookleftarrow$ ~~$\mathcal{W}(X_T) \Rightarrow$~~ $\mathrm{Rep}(T)$.

full subcat. spanned by fibers.

(Seidel: this has $\chi = -2$, if fatten, get genus 1 surface w/ 2 bd. components;

get an LG model, $\mathbb{C}^2$ rel thickened 

which is exactly the LG model for $x^2 + y^4$)

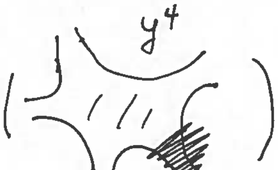
which is a stabilization of $(\mathbb{C}, y^3)$.!

(Rmk: there's a transverse knot/link, if you go around the boundary (2,4) torus

Rmk/question: (D. Treumann)

$\mathcal{W}(X_{\bullet \to \bullet \to \cdots})$ high dimension

$\mathrm{Rep}(\bullet \to \bullet \to \cdots)$

$\mathcal{W}($  $)$ y^4

low dimension.

Any explanations? "taking a slice"?

(Paul: per.($\mathbb{CP}^n$, std. struct) "stabilization",

12 sth assoc to A_n surface singularity;

$\exists$ known relation between topology increasing vs. dimension going up).

Question: what if T is not an ADE tree? Can one reduce the dimension to something low? (A trees can be reduced to dim 2; have 2nd specializations)

DE have 2d specializations)

conj: can always be brought down to $\mathbb{C}^2$??

There's a category

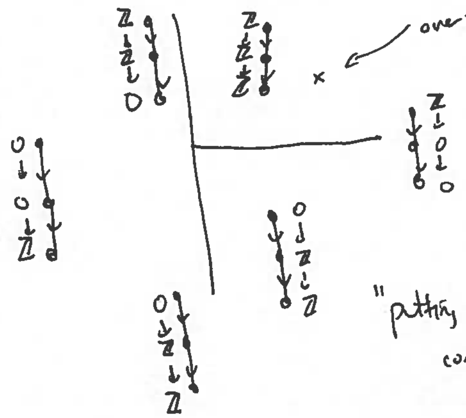
"Trees": objects are rooted trees

morphisms $T \dashrightarrow T'$ are correspondences
$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[d, "q"] & T' \\ T & \end{tikzcd}$$

Both X_T and $DRep(T)$ are functors on "Trees"

(here we're abusing notation & thinking of X_T via stabilizing).

Ex:



over this point, can take a Lag'n cotangent fiber; this should correspond to the corresponding representation listed. (perhaps the "indecomposables").

"putting $\mathbb{Z}$ in component, 0's elsewhere)

$$\begin{tikzcd} \boxed{Rep(T'')} \arrow[r] \arrow[d] & Rep(T) \\ {} & \end{tikzcd}$$

gives a distinguished class of correspondences, from $Rep(\bullet)$ "subtrees." → or specific Lag'n.

ex:

$$\mathcal{W}(X_* = T^*([0,1])^k) = \mathcal{W}(T^* B^k) = Rep(\bullet)$$

(top dimensional strata

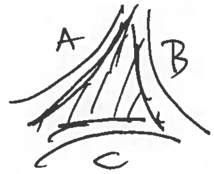
~~Bothers~~ & also have correspondences coming from

$\bullet \to\to \bullet \xhookrightarrow{\text{inject}}$

$\|$; should give exact triangles in the category).

$\bullet \to \bullet$

(Recall $X_{\bullet\to\bullet}$ =



& $\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, "{[1]}"] & \end{tikzcd}$.

any any tree T has many correspondences from $X_{\bullet\to\bullet}$ (say by inclusion): allows us to detect exact triangles in $\mathcal{W}(X)$ by embedding in $X_{\bullet\to\bullet}$ (come from correspondences).

Cor (of prop?) functor ~~trees~~ Trees $\to \mathcal{W}(X_{(\cdot)})$ are fully faithful

(b/c true on representable trees).

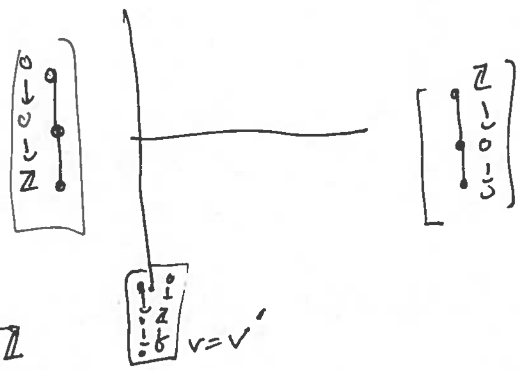
Sketch of Proof of Prop:

Set $\mathcal{W}_T := \mathcal{W}(X_T)$

$\cup$I

S_T = Lag'ns corresponding to rep's supported on a single vertex.

ex:



$$\mathrm{Hom}(S_v, S_{v'}) = \begin{cases} \mathbb{Z} & v = v' \\ \mathbb{Z}[-1] & v \to v' \\ 0 & \text{else} \end{cases}.$$

In fact, it turns out that $H^\circ S_T$ is intrinsically formal, so $H^\circ S_T \dashrightarrow S_T$. (very few morphisms)

To prove the result, it suffices to show S_T generates other fibers in $\mathcal{W}_T$.

Can do this by using functoriality.

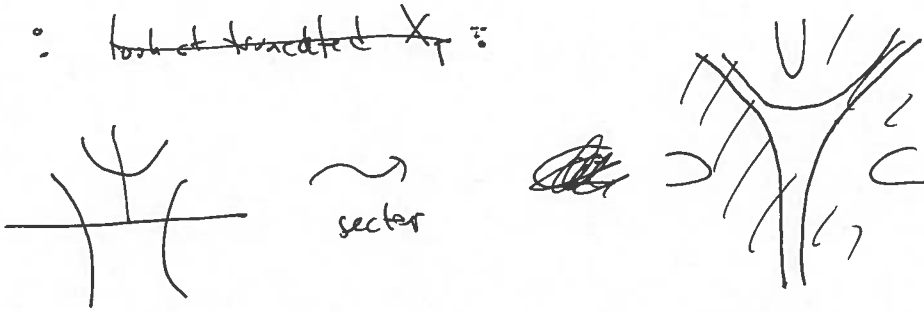
Conjecture: $\exists$ a cutoff Reeb vectorfield on $\partial_\infty X_T$ with no closed orbits.

(if true, then $\mathcal{OC}: HH_{*-n}(\mathcal{W}_T) \longrightarrow SH^\circ(X_T, \partial X_T)$ is an isomorphism

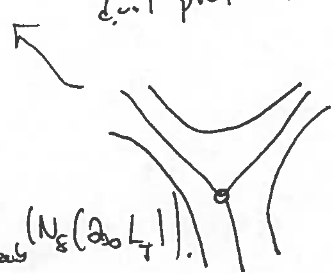
$\Rightarrow$ Any Weinstein manifold w/ arboreal spine satisfies generation criterion.).

Rmks here: ~~look at truncated X_T~~:

Ex:



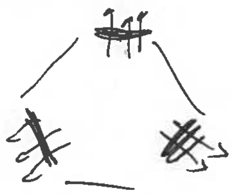
dual picture:



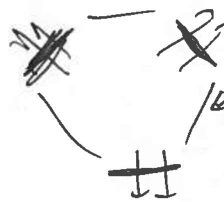
idea: $\partial_\infty L_T \subseteq S^{2n-1}$

claim: $S^{2n-1} \setminus N_\varepsilon(\partial_\infty L_T) \simeq F\ell^{\varepsilon}_{\mathrm{Reeb}}(N_\varepsilon(\partial_\infty L_T))$.

Ex: truncated X_T :



vs.



But the condition of being an arboreal sector means char foliation has no orbits; $\Rightarrow$

has no orbits.