10/26/2016 – H. Ruddat, Logarithmic Gromov-Witten Invariants I

old fact: unique conic through 5 points

Q: how many conics are tangent to five given conics?

$\rightsquigarrow$ ans: 3264.

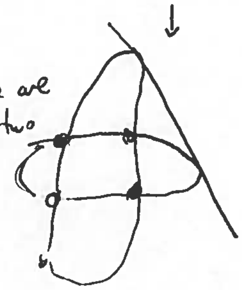
[Chasle – Halphen – Schubert]: any enumerative question about families of conics can be answered by knowing point & tangency to line numbers (these are much simpler questions).

E.g., for Q above, need to know, plus also:

	tangent to lines	0	1	2	3	4	5
(*)	points	5	4	3	2	1	0
	# conics	1	2	4	4	2	1

symmetric by (projective) duality

here are the two



So, we care about tangency conditions.

Problem: how to ensure invariance to placement of points or lines?

Answer: Gromov-Witten theory:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{univ. curve}} & X \\ (**)\ \downarrow \uparrow \sigma_i & \nearrow ev_i & \\ \mathcal{M} & & \end{array}$$

σ_i marked points (sections of univ. curve)

$\mathcal{M}$ moduli of stable maps

Here, $X = \mathbb{CP}^2$.

How to incorporate tangencies? $L \subseteq X = \mathbb{CP}^2$ line, pick a section of $T^*X|_L$ perpendicular to TL.

Pull back to $\mathcal{M}$ under ev. map, to get tangency condition (ψ-class).

Problem: $\langle \underbrace{pt, \ldots, pt}_{4 \text{ pt conditions}}, \psi(\ell) \rangle_{\mathbb{CP}^2, 2\ell} = 1$ (but we wanted 2!)

$= 2 - 1$ ← actually, the section had a pole; so need to compensate for it!

Rmk: Usual way to define ψ classes involves Chern classes of fibrewise tangent bundles of (**), but can check these give the same things using rel'n between chern class & sections of line bundles.

Better/other solution: relative stable maps (Ionel, Li)

$(C, p_1, \ldots, p_s) \longrightarrow (X, D)$

X smooth

D smooth divisor

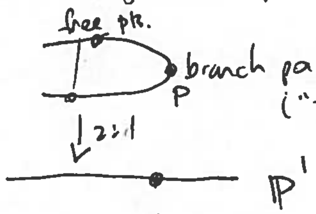
$p_i \longmapsto D$

with prescribed tangencies

(+ other additional marked points as desired w/o constraints)

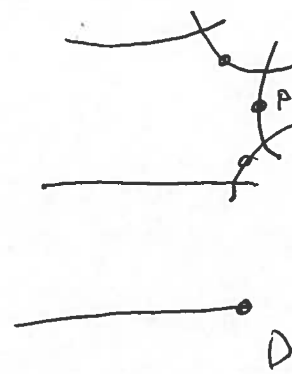
Problem now: How to get compact moduli space?

e.g.



("tangency order 2" constraints)

this moduli space is 3-d, (branch point, + 2 other points) & can degenerate:

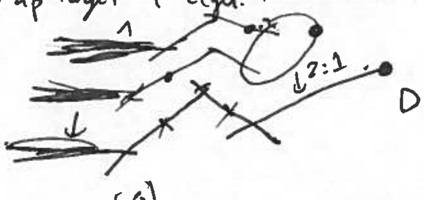


this contract component maps entirely to D, so tangency/branch order information gets lost geometrically

Two solutions:

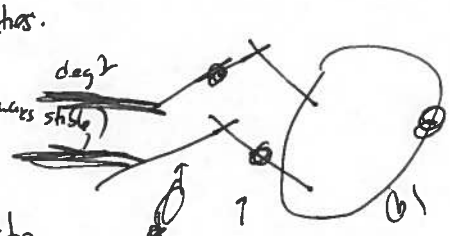
Li: "expanded degenerations":

blow up target along space of maps ("degen. to normal cone"): replace $\mathbb{P}^1$ by



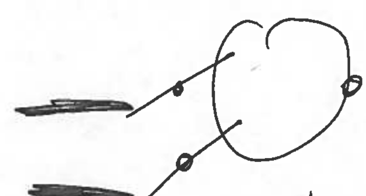
Each given curve lands in a finite degeneration (b/c marked pts. "used up"), but the technique allows for all arbitrarily ~~definitely~~ large degenerations

other possibilities.



deg 0 in neck (so each generation needs a marked point) to fix $\mathbb{C}^*$ ambiguity/ stabilize.)

&

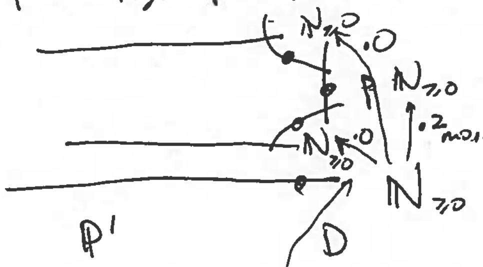


here as
($\mathbb{P}^1$ with w/ (a), (b) boundary).

$\rightsquigarrow$ get a nice moduli problem, w/ no loss of order of tangency.

Siebert: keep the target space fixed:

Gross, Chen, Abramovich



Add monoids, ~~[illegible]~~ ~~mult. by zero~~.

"log stable map."

(formally keep track of multiplicities 2, 0, 0, w/ imposition that they converge to geom. multiplicities when non-constant)

There's a natural map from the

Li compactification $\longrightarrow$ Siebert one which contracts the $\mathbb{P}^1$ worth of solutions.

Observation: $\mathcal{M}_{rel} \longrightarrow \mathcal{M}_{log}$ [Abramovich, Marcus, –] the VFCs pull back nicely, so get same numbers. (for a smooth divisor).

birational

here: contracts a $\mathbb{P}^1$.

Log geometry

$\{xy = t\}$ fam. of nodal curves

$\phantom{\{}\Vert$

X_t, then X_0 is singular in usual alg. geometry.

$xy \downarrow$

$\mathbb{C}$

(to see this, compute sheaf of rel. diff'l forms:

$$\left(\frac{k[x,y] \cdot dx}{xy} \oplus \frac{k[x,y]\, dy}{xy} \right) \Big/ dt$$

to make relative the base

$\cancel{dt}\ dt = d(xy)$
$= x\,dy + y\,dx$

this has a 2D stalk at $\vec{0}$, hence is not a loc. free sheaf.

$\Omega_{X_0/0}$ is not locally free.

Now consider this with log differential forms (not yet defined) w.r.t. log structures:

$$\Omega_{X_0^+/0^+} = \left(\frac{k[x,y]}{xy} \frac{dx}{x} \oplus \frac{k[x,y]}{xy} \frac{dy}{y} \right) \Big/ \frac{dt}{t}$$

(replace dx, dt, etc. by $d\log x$, $d\log t$, etc.)

now: $\frac{dt}{t} = \frac{dx}{x} + \frac{dy}{y}$

$\rightsquigarrow$ is a locally free sheaf.

$\Rightarrow X_0$ is log smooth.

Log geom. is a tool to treat log diff'l forms functorially (& have relative notions).

What is a log scheme

Def: a) A pre-log structure is a triple $(X, \mathcal{M}, \alpha)$, where

- X scheme
- $\mathcal{M} \xrightarrow{\alpha} (\mathcal{O}_X, \cdot)$ ← the relevant monoid structure

$\mathcal{M}$: sheaf of monoids on X; α: monoid map

(often in étale topology, but Zariski topology makes sense too).

b) Pre-log structure is a log structure iff

$$\alpha\big|_{\alpha^{-1}(\mathcal{O}_X^*)} \xrightarrow[\alpha]{\sim} \mathcal{O}_X^* \leftarrow \text{invertible elements}$$

(so $\mathcal{M}$ has "no new invertible functions as target") $(\Rightarrow \mathcal{M}_X^* = \mathcal{O}_X^*)$

Examples: X scheme, $D \subseteq X$ pure codim 1 subset, then

$$\mathcal{M}_{(X,D)} := j_* \mathcal{O}^*_{X \setminus D} \cap \mathcal{O}_X \overset{\alpha}{\hookrightarrow} \mathcal{O}_X \text{ is a log structure.}$$

" invertible fns on $X \setminus D$, that don't have poles when extended "

$j : X \setminus D \hookrightarrow X$

b) P monoid (finitely generated)

$$P \longrightarrow \mathbb{Z}[P] = \left\{ \sum a_i z^{p_i},\ \underset{p_i \in P}{a_i \in \mathbb{Z}} \right\}$$ "monoid ring"

$$p \longmapsto z^p$$

"Specifying this" get

$$\rightsquigarrow \underline{P} \longrightarrow \mathcal{O}_{\operatorname{Spec} \mathbb{Z}[P]}$$ is a pre-log structure

or $\mathbb{K}[P]$.

(b/c typically, say if we used $\mathbb{K}[P]$, P doesn't know about invertibles in $\mathbb{K}$).

Construction Lemma:

$(\mathcal{M}, \alpha, X)$ pre-log structure. Then define the associated log structure

$$\mathcal{M}^a := \frac{\mathcal{M} \oplus \mathcal{O}_X^*}{\{(m, \alpha(m)^{-1}) / m \in \alpha^{-1}(\mathcal{O}_X^*)\}}.$$ analogy: like "sheafification".

$$\alpha^a \downarrow$$
$$\mathcal{O}_X$$

w/ $\alpha^a(h, x) = \alpha^a(h) \cdot x$.

Example: $P = \sigma^\vee \cap M$ (Fulton) toric monoid.

dual cone, ~~to~~ dual lattice

$$X = \operatorname{Spec} \mathbb{C}[P] \quad (\text{or } \mathbb{Z}[P]$$

$\cup\!|$ affine toric variety.

D

boundary divisor.

Then, $(P \to \mathbb{C}[P])^a = \mathcal{M}_{(\operatorname{Spec} \mathbb{C}[P], D)}$.

$X \xrightarrow{f} Y$ map of schemes, & $(\mathcal{M}, \alpha, Y)$ log structure on Y;

then, have a pull-back log structure on X:

$$\mathcal{M} \xrightarrow{\alpha} \mathcal{O}_Y$$

pull back:

$$\left(f^{-1}\mathcal{M} \xrightarrow{\alpha} f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X\right)^a =: \mathcal{M}_X .$$

Makes the process functorial [Illusie's insight].

Def: $(X, \mathcal{M})$ log. scheme is fs (fine & saturated) if locally on X there are charts for the log structure. (point: monoid sheaves in general can be fairly nasty; this is a nice set of such)

i.e., on each open cover, $\exists$ P "finitely generated" "integral" "saturated" on open sets (some very nice monoid)

w/ $\mathcal{M} = (P \longrightarrow \mathcal{O}_X)^a$ locally. not defined, but "nice" condition.

(here, might be better to use the étale topology, esp. for divisors w/ self-intersections)

Def: $(X, \mathcal{M}) \longrightarrow (Y, \mathcal{N})$ map of fs log schemes is log smooth iff, étale locally,

$$\begin{tikzcd} P \arrow[r] & \mathcal{O}_X \\ Q \arrow[r] \arrow[u, hook] & \mathcal{O}_Y \arrow[u] \end{tikzcd}$$

(a) $\boxed{\text{injection}}$ "pullback" "$\Leftrightarrow$" "flatness") $\leftarrow$ "pull back" technically

and

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left] \arrow[dr] & Y \times_Q P \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{C}[P] \arrow[d] \\ & Y \arrow[r] & \operatorname{Spec} \mathbb{C}[Q] \end{tikzcd}$$

(b) $\boxed{\text{is smooth}}$

flat (a map of toric varieties) (by injectivity)

(locally defined map only)

So a log smooth map is locally "a composition of a smooth map w/ (pullback of) toric map".

cf. [Kato, toroidal characterization of smoothness]

There's another usual definition "formal smoothness", "on lifting log tangent vectors", equivalent to this one.

Rmk: Given $D \underset{i}{\subseteq} X$, can pull back $i^* \mathcal{M}_{(X,D)}$. Tells you how D sits in X, which is crucial to GS program.